



UGA High School Varsity Math Tournament
October 25, 2025

WITH SOLUTIONS

WRITTEN TEST

TIME: 90 MINUTES

LENGTH: 25 PROBLEMS

MAX SCORE 250 POINTS

10 POINTS FOR A CORRECT ANSWER,

0 POINTS FOR AN INCORRECT ANSWER,

AND 2 POINTS FOR AN ANSWER LEFT BLANK.

Problem 1. On this exam, you will receive 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank. What is the total number of possible overall scores?

(A) 100

(B) 110

(C) 115

(D) ♥120

(E) 125

Solution. Let c be the number of correct answers and w the number of wrong answers. We have $0 \leq c, w$ and $0 \leq c + w \leq 25$. The associated score is

$$10c + 0w + 2(25 - c - w) = 50 + 8c - 2w = 50 + 2(4c - w).$$

It is enough to determine the number of possible values of $4c - w$ under the above constraints. The bounds for the expression are $-25 \leq 4c - w \leq 100$ however not all these values are attained.

Call R_c the range of values for a fixed c . For example, $R_0 = \{-25, -24, \dots, 0\}$, $R_1 = \{-20, -19, \dots, 4\}$, etc. Note that for a fixed c , the values in R_c are “convex”, i.e. if $x, y \in R_c$ and $z \in \mathbb{N}$ is such that $x \leq z \leq y$, then $z \in R_c$. More precisely, $R_c = [4c - (25 - c), 4c] \cap \mathbb{N} = [5c - 25, 4c] \cap \mathbb{N}$.

Two successive sets R_c and R_{c+1} form a continuum iff $4c \geq 5(c+1) - 25 - 1$ iff $c \leq 21$. This means that the intervals R_0 to R_{22} form the set $[-25, 88]$. Thus, if there are any gaps, they are between 89 and 100. The simplest way to identify these gaps is to tabulate the intervals R_{23} to R_{25} , these are $[90, 92] \cap \mathbb{N}$, $[95, 96] \cap \mathbb{N}$ and $[100, 100]$. The gaps are therefore 89, 93, 94, 97, 98 and 99. There are thus $126 - 6 = 120$ possible scores.

Problem 2. The underbrace notation means that a sequence of digits is repeated. For example, $5 \underbrace{28}_5 67 = 5, 282, 828, 282, 867$ and $1, \underbrace{272}_2 = 12, 721, 272$. If we define

$$\Omega = \underbrace{\underbrace{123}_{123} \underbrace{456}_{456} \underbrace{789}_{789}}_{123456789},$$

what is the remainder of the division of Ω by 11?

- (A) 4 (B) 5 (C) 6  (D) 7 (E) 8

Solution. Recall that a number $a = a_n \dots a_2 a_1 a_0$, where a_i denotes a digit, is congruent to $\sigma(a) = a_0 - a_1 + a_2 - \dots + (-1)^n a_n$ modulo 11. E.g. the remainder of the division of 135 by 11 is $5 - 3 + 1 = 3$.

More generally, we then have have $\underbrace{a_n \dots a_2 a_1 a_0}_N$, which is congruent to

$$\sum_{i=1}^N (-1)^{i+1} \sigma(a) = \begin{cases} 0, & N \text{ even,} \\ \sigma(a), & N \text{ odd} \end{cases}$$

when n is even and congruent to

$$N\sigma(a)$$

when n is odd.

Lastly, $10^n a$ is congruent to $(-1)^n a$ modulo 11.

Combining the above properties, we get

$$\begin{aligned} \Omega &= \underbrace{\underbrace{123}_{123} \underbrace{456}_{456} \underbrace{789}_{789}}_{123456789} \\ &= \underbrace{789}_{789} + 10^{3 \cdot 789} \underbrace{456}_{456} + 10^{3 \cdot (456 + 789)} \underbrace{123}_{123} \\ &= \sigma(789) - \sigma(123) \\ &= 9 - 8 + 7 - 3 + 2 - 1 \\ &= 6 \pmod{11} \end{aligned}$$

Problem 3. Any line in Euclidean space can be described as the set of points of the form

$$(a + \alpha t, b + \beta t, c + \gamma t)$$

where $a, b, c, \alpha, \beta, \gamma$ are constants, not all of α, β and γ are 0 and t varies over $\mathbb{R}$. John traces a line on his computer and realizes that all points $P(x, y, z)$ on it satisfy two properties: (i) the coordinates seem to satisfy some Fibonacci like rule, i.e. $x + y = z$ and (ii) the middle coordinate is always the average of the extreme coordinates, i.e. $y = \frac{x+z}{2}$. Using the notation above, what is $\frac{\beta+\gamma}{\alpha}$?

- (A) 0 (B) 2 (C) $\frac{5}{2}$ (D) $\frac{1+\sqrt{5}}{2}$ (E)  5

Solution. The constraints $x + y = z$ and $x + z = 2y$ imply that $y = 2x$ and $z = 3x$. Replacing x with $a + \alpha t$, etc., we get that $b + \beta t = 2a + 2\alpha t$ and $c + \gamma t = 3a + 3\alpha t$. Since this relation must hold for all values of t , we can deduce that $\beta = 2\alpha$ and $\gamma = 3\alpha$. So that $\frac{\beta+\gamma}{\alpha} = \frac{5\alpha}{\alpha} = 5$.

Problem 4. For the New Year, the teacher copies all numbers from 1 to 2025 on a large blackboard. He tells the students that they are allowed to erase any two numbers provided that they write on the board the result of the expression

$$a \times b + a + b$$

where a and b are the two numbers erased. Notice that by doing so, the total amount of numbers on the boards decreases by one.

Alice begins the process by erasing 10 and 20 which she replaces by 230. The other students take turns repeating this procedure until only a single number remains. What is that number?

- (A) 2025×2026 . (B) 2026^{1013} . (C)  $2026! - 1$.
(D) $\frac{4050!}{2025!}$ (E) The number is variable.

Solution.

If $x_1, \dots, x_N$ are written on the board, consider the product $(x_1 + 1) \dots (x_N + 1)$. We claim that this product stays constant throughout the replacement process. Indeed, observe that at a given step, a and b are replaced with $(a + 1)(b + 1) - 1$.

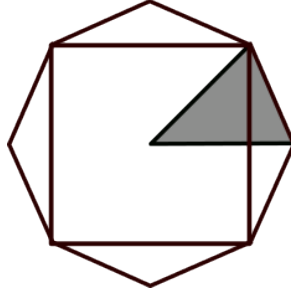
Therefore, when there is only one number, it can only be such that adding 1 to it equals the product that we have at the beginning, thus

$$(1 + 1)(2 + 1)(3 + 1) \dots (2025 + 1) - 1 = 2026! - 1.$$

Problem 5. A regular octagon is built so that four of its corners coincide with the four corners of a square of area 1. What is the area of the octagon?

- (A) $\sqrt{2}$ ♡ (B) $\frac{3\sqrt{2}}{4}$ (C) $4\sqrt{2} - 4$ (D) $2\sqrt{3} - \sqrt{2}$ (E) Some other constant.

Solution. The octagon and square are depicted in the image below.



To determine the area of the octagon, it is sufficient to compute the area of the shaded triangle which is 1/8th of the total figure. By regularity, this is an isosceles triangle; its equal sides have length $\frac{\sqrt{2}}{2}$ – half of the square’s diagonal, and they form an angle of 45° . The area of the triangle is thus

$$\frac{\frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \sin(45^\circ)}{2} = \frac{\sqrt{2}}{8}$$

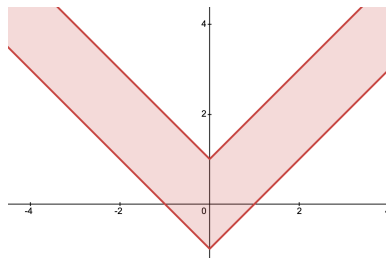
and the total area of the octagon is $\sqrt{2}$.

Problem 6. What is the radius of the largest circle included in the region of the plane defined by

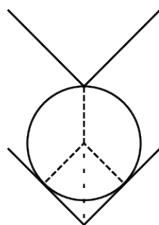
$$|x + |y|| \leq 1$$

- (A) $\frac{1}{\sqrt{2}} + 1$ (B) $\sqrt{2} - 1$ (C) $\frac{\sqrt{2}}{3}$ (D) $1 - \frac{1}{\sqrt{2}}$ (E) $2\sqrt{2} - 2$ ♡

Solution. Drawing the region, we find by symmetry the center of the circle must be along the axis and pass through the “cusp” point (0,1).



Here is a closeup view of the circle.



Using the fact the circle ought to be tangent to the lines $y = -1 + x$ and $y = -1 - x$ to guarantee maximality, we see that the separation between the two cusps is $2 = r + r\sqrt{2}$ where r is the radius of the circle. From there,

$$r = \frac{2}{1 + \sqrt{2}}$$

which after multiplication with the conjugate of the numerator gives the required result $\Rightarrow r = 2\sqrt{2} - 2$.

Problem 7. Scientists discovered a small planet with radius 30 km such that its surface is entirely covered with water. A spherical meteorite hits this planet and fully sinks in the water. The height of the water increased by 1 cm everywhere. What is the estimated radius of the falling orb?

- (A) $\sqrt[3]{9} \times 100 \text{ m}$ (B) 100 m (C) 3 km (D) 300 m ♡ (E) 1 km

Solution. The volume of the meteorite is equal to the amount of displaced water which is, in centimeters,

$$\frac{4}{3}\pi r^3 \approx 4\pi(30 \times 10^5)^2 \times 1$$

So $r^3 \approx 27 \times 10^{12}$ and $r \approx 3 \times 10^4 \text{ cm} = 300 \text{ m}$.

Problem 8. A robot begins a journey from the corner of a rectangular room RUME with a beginning trajectory of 45° . The robot collides with one of the two longer walls of the room at point P, turns 90° to the right, and continues ricocheting from the walls at A and T, turning 90° each time. The robot then travels until it intersects its own path at the point H (as shown in the diagram, though not to scale). At the end of its journey, the robot has enclosed the rectangle PATH, which is similar to the rectangle RUME. What is the ratio of the room's longer wall to its shorter wall?

- (A) $\sqrt{2}$ (B) $\varphi = \frac{1+\sqrt{5}}{2}$ ♡ (C) e (D) π (E) $2 - \sqrt{2}$

Solution.

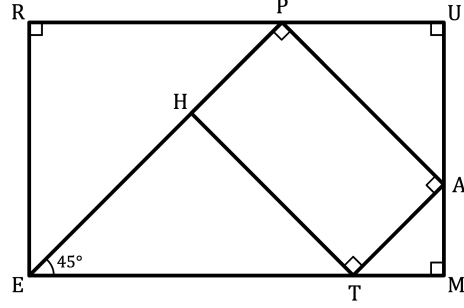


Figure 1: The journey of a robot.

Because we only need the ratio of the side lengths of the rectangle $RUME$, we can let $|RE| = 1$ without loss of generality.

For ease of notation, let

$$|ME| = x, \quad |PA| = p, \quad |AT| = q.$$

Since $PATH \sim RUME$, we have

$$\frac{p}{q} = x. \tag{1}$$

Since $\triangle PRE$ is a 45-45-90 right triangle and $|RE| = 1$, we get $|PR| = 1$ as well. Hence,

$$|UP| = x - 1.$$

Since $\triangle UAP$ is also a 45-45-90 right triangle and $|UP| = x - 1$, we obtain

$$|AU| = x - 1.$$

Therefore,

$$|AM| = |UM| - |AU| = 1 - (x - 1) = 2 - x.$$

Now, since both $\triangle AUP$ and $\triangle MAT$ are 45-45-90 triangles, we have

$$\frac{p}{q} = \frac{|UP|}{|AM|} = \frac{x - 1}{2 - x}.$$

By (1), it follows that

$$x = \frac{x - 1}{2 - x}.$$

Multiplying both sides by $(2 - x)$,

$$2x - x^2 = x - 1.$$

Simplifying,

$$0 = x^2 - x - 1.$$

The positive root of this quadratic is

$$x = \frac{1 + \sqrt{5}}{2} = \varphi.$$

Problem 9. We call a date Schur if the sum of the month number and the day of the month equals the last two digits of the year. For example, October 15, 2025 is Schur because $10+15=25$. Let N be the number of Schur dates that occur between January 1, 2000 and December 31, 2999? What is the sum of the digits of N .

- (A) 7 (B) 21 (C) 13 (D) 14 (E) 16

Solution. Notice a leap day can never be a Schur date, because $2 + 29 = 31$ is never divisible by 4. Because we are considering only the last two digits of the year, it suffices to find the number of Schur dates between January 1, 2000 and December 31, 2099, and then multiply the result by 10.

For every valid month, day pair on the calendar (m, d) , there is exactly one year in a given century whose last two digits equals $m + d$. Here, we are using the fact that $m + d \leq 43 < 99$. Thus, the number of Schur dates in a century is the number of possible choices of (m, d) , which is precisely 365.

Thus, $N = 365 \times 10 = 3650$, so the answer is 14.

Problem 10. Consider the infinite table below.

4	7	10	13	16	...
7	12	17	22	27	...
10	17	24	31	38	...
13	22	31	40	49	...
16	27	38	49	60	...
19	32	45	58	71	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

It is built as follows: the top left entry is 4, the first column is the transpose of the first row, each row is an arithmetic progression, the first row has an increment of 3 and each row has increment of two more than the previous row.

If you start from number 4, go down one step to 7, go to the right one step to 12, go down to 17, etc. repeat those steps until you have moved 100 times down and 100 times to the right. What entry will you read in the corresponding box?

- (A) 5,151 (B) 10,100 (C) 20,000 (D) 20,200 (E) 20,604

Solution. Call Δ_i the i -th diagonal entry, i.e. the entry on the i th row and column. We are looking for Δ_{101} . Notice that step in row or column i is $2i + 1$. We have

$$\Delta_{i+1} = \Delta_i + (2i + 1) + (2i + 3) = \Delta_i + 4i + 4.$$

Let's introduce the notation $\delta_i = \frac{\Delta_i}{4}$ which yields the recursive formula

$$\delta_{i+1} = \delta_i + i + 1.$$

Since $\delta_1 = 1$, we recognize the recursive formula for triangular numbers,

$$\delta_i = \sum_{k=1}^i k = \frac{i(i+1)}{2}.$$

and therefore $\Delta_{101} = 4\delta_{101} = 2 \cdot 101 \cdot 102 = 20604$.

Source: Inspired by the OMB

Problem 11. What is the sum of the first 3 digits after the period in the decimal expansion of $\sqrt{2026}$?

- (A) 2  (B) 5 (C) 7 (D) 11 (E) 19

Solution. Note that $\left(a + \frac{b}{2a}\right)^2 = a^2 + b + \left(\frac{b}{2a}\right)^2$. If b is relatively small compared to a , the term $\left(\frac{b}{2a}\right)^2$ will also be small. In our case, if we let $a = 45$ and $b = 1$, we get that $2026 = 45^2 + 1 + \left(\frac{1}{90}\right)^2$, i.e.

So we can approximate $\sqrt{2026}$ by $45 + \frac{1}{90} = 45.01111$. Since $\left(45 + \frac{1}{90}\right)^2 > 2026$ and $45.011^2 = 2,025.9\dots$ we are certain that this is a sufficient approximation. The sum is thus $0 + 1 + 1 = 2$.

Problem 12. Harry writes on a piece of paper the result of

$$\frac{1000}{2.014 - 1.007^2}.$$

Unfortunately, he leaves the paper in his pocket and all he can read after it goes through the washing machine is

1000.???00240111...

The three first digits after the period have become non legible; what is the sum these three digits?

- (A) 0 (B) 7 (C) 8 (D) 13  (E) 15

Solution. The computation is the evaluation at 1.007 of

$$\frac{1000}{2x - x^2} = \frac{1000}{1 - (x - 1)^2} = 1000 \cdot (1 + (x - 1)^2 + (x - 1)^4 + (x - 1)^6 + \dots).$$

The latter can easily be evaluated to $1000 \cdot (1 + 0.000049 + \dots) = 1000.04900\dots$

Problem 13. A *Lissajous curve* is a plane curve defined parametrically by

$$x(t) = \sin(at + \delta), \quad y(t) = \sin(bt),$$

In other words, to each real number t we plot the point of coordinates $(x(t), y(t))$ in the plane.

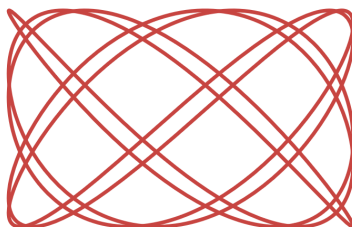


Figure 2: The Lissajous curve $(\sin(5t), \sin(7t + 1))$.

At how many points does the Lissajous curve $(\sin 4t, \sin 3t)$ intersect itself?

- (A) 9 (B) 12 (C) 16 (D) 17  (E) 25

Solution. Up to rescaling, points of intersection correspond to non trivial solutions of the system

$$\begin{cases} \sin(8\pi t) = \sin(8\pi s), \\ \sin(6\pi t) = \sin(6\pi s). \end{cases}$$

This is the same curve but “drawn” at a higher frequency. The period is now 1. Henceforth, we shall assume that $0 \leq s < t < 1$ to avoid double counting. This system translates to

$$\begin{cases} 8t = 8s + 2k \text{ or } 8t = 1 - 8s + 2k, \\ 6t = 6s + 2k \text{ or } 6t = 1 - 6s + 2k. \end{cases}$$

or

$$\begin{cases} t = s + \frac{k}{4} \text{ or } t = \frac{1}{8} - s + \frac{k}{4}, \\ t = s + \frac{1}{3}k \text{ or } t = \frac{1}{6} - s + \frac{1}{3}k. \end{cases}$$

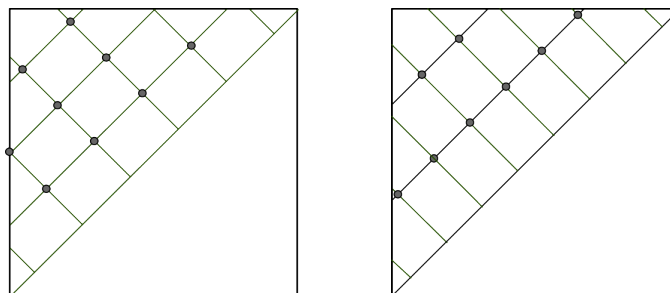
Points of self-intersection of the curve correspond, to the points of intersection of these lines, taken pairwise, within the unit square $[0, 1) \times [0, 1)$. Excluding parallel lines, we have two systems to look at:

$$\begin{cases} t = s + \frac{k}{4} \\ t = \frac{1}{6} - s + \frac{l}{3}. \end{cases}$$

and

$$\begin{cases} t = \frac{1}{8} - s + \frac{k}{4}, \\ t = s + \frac{l}{3} \end{cases}$$

One could solve these systems algebraically but it is just as fast to sketch them. One easily gets the following two diagrams:



Where it is easy to identify $9 + 8 = 17$ points of intersection.

Problem 14. Is the number $2024^5 + 2025$ prime?

- (A) Yes
- (B) No, it is a multiple of 3
- (C) No, it is a multiple of 5
- (D) No, it is a multiple of 11
- (E) No, it is a multiple of another prime



Solution. The key is to observe that the polynomial $X^5 + X + 1 = X^3 X^2 + X + 1$ hence it has the cubic roots of unity as roots. This suggest dividing it by $X^2 + X + 1$ to realize the factorization $X^5 + X + 1 = (X^2 + X + 1)(X^3 - X^2 + 1)$. Plugging in $X = 2024$, shows that $2024^5 + 2025 = 2024^5 + 2024 + 1$ is composite since $2024^2 + 2024 + 1 > 1$ and $2024^3 - 2024^2 + 1 = 2024^2 \cdot 2023 + 1 > 1$. Moreover, it is neither a multiple of 3 nor of 5 as 2025 is but not 2024. It is not a multiple of 11 as 2024 is multiple of 11 but not 2025.

Problem 15. A robot moves in a rectangular room and everytime he hits a wall, he bounces off, so that the angle of incidence equals the angle of reflection, and continues his expedition. This time, the robot begins its journey on the long side 3 feet away from the corner with an initial angle of 45° with the wall, and away from that corner – as is depicted below. How long will it take before he gets back to his initial position if the dimensions of the room are 35 feet by 14 feet?

- (A) 140
- (B) $105\sqrt{2}$
- (C) $140\sqrt{2}$
- (D) 210
- (E) 377

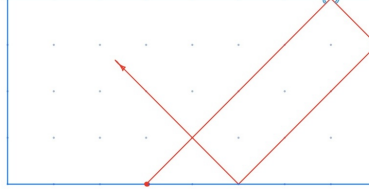
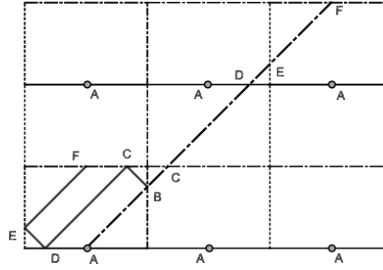


Figure 4: Another robot.

Solution. The motion of the robot can be modelled by a straight line on a tessellation of the plane by copies of the room, as depicted below:



Assuming that we place the origin at the bottom left corner of the table so that our initial point has coordinates $(3, 0)$, the virtual images of this point will have coordinates $(70k \pm 3, 28m)$ where k is even and m is any integer. We are looking for points in the first quadrant that lies on the line $y = x - 3$, i.e. we want to solve

$$70k \pm 3 = 28m + 3$$

in $\mathbb{N}^+$. Consider the two possible choices: $70k - 3 = 28m + 3$ or $70k + 3 = 28m + 3$. The former choice has no solution as can be seen by reducing modulo 7. The latter reduces to $10k = 4m$ and, since k is even, $5k' = m$ where $k = 2k'$. I.e. solutions are of the form $(140k' + 3, 140k')$ and the smallest choices is $(143, 140)$. The distance between that point and $(3, 0)$ is $140\sqrt{2}$.

N.B. One can easily generalize this problem and realize that the answer is $\sqrt{2}$ times the lcm of the length of the sides.

Problem 16. Let $\lfloor x \rfloor$ be the integer nearest to x , with the convention that if we round up if the fractional part of x is $\frac{1}{2}$. E.g. $\lfloor 4.87 \rfloor = 5$, $\lfloor 4.33 \rfloor = 4$ and $\lfloor 4.5 \rfloor = 5$. Evaluate the following sum:

$$\frac{1}{\lfloor \sqrt{1} \rfloor} + \frac{1}{\lfloor \sqrt{2} \rfloor} + \frac{1}{\lfloor \sqrt{3} \rfloor} + \cdots + \frac{1}{\lfloor \sqrt{2024} \rfloor} + \frac{1}{\lfloor \sqrt{2025} \rfloor}$$

(A) 84

(B) 87

(C) $\frac{191}{2}$

(D) 89 

(E) 90

Solution. The equality $\lfloor \sqrt{k} \rfloor = n$ is equivalent to $n - \frac{1}{2} \leq \sqrt{k} < n + \frac{1}{2}$ or, after squaring,

$$n^2 - n + \frac{1}{4} \leq k < n^2 + n + \frac{1}{4}.$$

This is an interval of width $(n^2 + n + \frac{1}{4}) - (n^2 - n + \frac{1}{4}) = 2n$ and since it is half-open, it always contains $2n$ distinct integers. More precisely, the values of k can be split in two regions: $k_1 < k_2 < \dots < k_n = n^2 < k_{n+1} < \dots < k_{2n}$. Thenceforth, the above sum $\frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \dots$ can actually be grouped as follows:

$$\underbrace{\frac{1}{1} + \frac{1}{1}}_2 + \underbrace{\frac{1}{2} + \dots + \frac{1}{2}}_4 + \cdots + \underbrace{\frac{1}{n} + \dots + \frac{1}{n}}_{2n} + \cdots + \underbrace{\frac{1}{45} + \dots + \frac{1}{45}}_{45}$$

Or, more simply,

$$\left(\sum_{k=1}^{44} 2 \right) + \frac{1}{2} 2 = 89.$$

Problem 17. Evaluate

$$\sum_{n=1}^{2025} \# \left\{ (x, y) \in \mathbb{N}^2 \mid \frac{1}{x} + \frac{1}{y} + \frac{2^n - 1}{xy} = 1 \right\}.$$

where $\#$ denotes the number of elements in a set.

(A) 205,3350 

(B) 205,3351

(C) 205,420

(D) 212,314

(E) 212,420

Solution. The equation $\frac{1}{x} + \frac{1}{y} + \frac{2^n - 1}{xy} = 1$ can be normalized to

$$y + x + 2^n - 1 = xy$$

or

$$2^n - 1 = xy - x - y.$$

After adding 1 on both sides, we get

$$2^n = (x - 1)(y - 1).$$

In other words, we are trying to factor 2^n into the product of two positive integers. There are $n + 1$ ways to do so as $2^n = 2^i 2^{n-i}$ for $i = 0 \dots n$. In other words, the n th term of the above summation is $n + 1$ and the quantity we are trying to evaluate is

$$\sum_{n=1}^{2025} (n + 1) = \sum_{n=2}^{2026} n = \frac{2025(2 + 2026)}{2} = 2053350.$$

Problem 18. Let $M(x) = \lfloor \sqrt{x} \rfloor$ where $\lfloor x \rfloor$ denotes the *floor* function, i.e. the largest integer $n \leq x$. E.g. $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$ and $\lfloor -4.4 \rfloor = -5$. How many integer solutions does the equation

$$M(M(M(x))) = 4$$

have?

- (A) 325,089  (B) 325,090 (C) 325,091 (D) 325,092 (E) Some other number.

Solution. Let $M(x) = y$ and $M(M(x)) = M(y) = z$. The equality above,

$$M(z) = \lfloor \sqrt{z} \rfloor = 4,$$

is equivalent to

$$4 \leq \sqrt{z} < 5$$

or $4^2 \leq z < 5^2$. We now have

$$4^2 \leq \lfloor \sqrt{y} \rfloor < 5^2$$

or, more simply,

$$4^2 \leq \sqrt{y} < 5^2$$

and hence $4^4 \leq y < 5^4$. Finally, this reduces to

$$4^4 \leq \lfloor \sqrt{x} \rfloor < 5^4$$

or, as above, to

$$4^8 \leq x < 5^8.$$

The number of solutions is $5^8 - 4^8 = (5^4 - 4^4)(5^4 + 4^4) = 325089$.

Problem 19. Imagine a non degenerate triangle whose interior angles are α , β and γ and such that the tangent of these angles form a geometric sequence,

$$\tan \alpha = r \tan \beta = r^2 \tan \gamma$$

where $r \geq 1$. What is the largest possible value of $\tan^2(\gamma)$?

- (A) $\frac{1}{3}$ (B) 2 (C) $\frac{3}{2}$ (D) 3 (E) $\frac{7}{2}$

Solution. Let α , β and γ be the interior angle of a triangle; they satisfy the identity

$$\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \cdot \tan \beta \cdot \tan \gamma.$$

Indeed, $\tan \gamma = -\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{\tan \alpha \tan \beta - 1}$ which substituted in the above equality reduces it to a tautology.

In this problem, the above identity reduces to

$$1 + r + r^2 = r^3 \tan^2(\gamma)$$

as $\gamma > 0$, or,

$$\tan^2(\gamma) = \frac{1 + r + r^2}{r^3}.$$

The above fraction is strictly decreasing as r grows, hence the solution is when $r = 1$ and we are dealing with an equilateral triangle.

Problem 20. A point P is located at a distance of 4 from the center of a circle C of radius 2. A line m is drawn through P and through two points Q and R on C , distant of 2 from one another. Consider the circle C_Q which is tangent to C at Q and passes through P ; and the circle C_R which is tangent to C at R and passes through P . What is the difference, in absolute value, between the diameters of the circles C_Q and C_R ?

- (A) 2 (B) 3 (C) $2\sqrt{2}$ (D) 4 (E) $3\sqrt{2}$

Solution. This problem can be solved in many ways. Let's do it using coordinates. Assume that C is centered at the origin, i.e. $C : x^2 + y^2 = 4$ and up to rotation we can assume that Q and R are located at $(2 \cos \pi/3, 2 \sin \pi/3)$ and $(2 \cos 2\pi/3, 2 \sin 2\pi/3)$, i.e. $Q : (1, \sqrt{3})$ and $R : (-1, \sqrt{3})$.

The point P can be taken to be $P : (3, \sqrt{3})$.

A circle tangent to C at Q has its center on the line $y = \sqrt{3}x$, i.e. $C_Q(r_Q) : (x - a)^2 + (y - \sqrt{3}a)^2 = r_Q^2$ is tangent to C at Q and similarly, $C_R(r_R) : (x - b)^2 + (y + \sqrt{3}b)^2 = r_R^2$ is tangent to C at R . Since these circles pass through P , we get

$$(3 - a)^2 + (\sqrt{3} - \sqrt{3}a)^2 = r_Q^2$$

and

$$(3 - b)^2 + (\sqrt{3} + \sqrt{3}b)^2 = r_R^2.$$

Since these circles also pass through respectively Q and R we also get

$$(1 - a)^2 + (\sqrt{3} - \sqrt{3}a)^2 = r_Q^2$$

and

$$(-1 - b)^2 + (\sqrt{3} + \sqrt{3}b)^2 = r_R^2.$$

Subtracting the respective equations we get that

$$0 = (3 - a)^2 - (1 - a)^2 = 8 - 4a,$$

i.e. $a = 2$ and

$$0 = (3 - b)^2 - (-1 - b)^2 = 8 - 8b,$$

i.e. $b = 1$. It is then immediate that $r_Q = 2$ and $r_R = 4$ and the difference is 2.

N.B. This problem can be solved without the requirement that the two points Q and R on C are distant of 2. You can show that independently of their position, the difference between the radii is always 2, i.e. the initial radius!

Problem 21. Recall the table of Problem 10.

4	7	10	13	16	...
7	12	17	22	27	...
10	17	24	31	38	...
13	22	31	40	49	...
16	27	38	49	60	...
19	32	45	58	71	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

What is the smallest number strictly larger than 2025 not appearing in this table?

(A) 2028 

(B) 2031

(C) 2067

(D) 2068

(E) 2069

Solution. Numbers in row 1 are of the form $1 + 3c$, $c \geq 1$. The step in row r is $2r + 1$. Hence numbers in column c and row r are of the form $a_{c,r} = 1 + 3r + (c - 1)(2r + 1) = c + r + 2rc$. Consider the number

$$2a_{c,r} + 1 = 4rc + 2r + 2c + 1.$$

It can be factored as

$$(2c + 1)(2r + 1),$$

i.e. the product of two odd numbers, respectively larger than 3. Thus any entry a in the table is such that $2a + 1$ is an odd composite (i.e. not prime) number. The smallest number larger than 2025 is 2028. Indeed, $2 \cdot 2026 + 1 = 4053$ is a multiple of 3 and $2 \cdot 2027 + 1 = 4055$ is a multiple of 5 while $2 \cdot 2028 + 1 = 4057$ is prime. Inspired by OMB.

Problem 22. What is the coefficient of x^{60} in the polynomial

$$(x-1)(x^2-1)(x^3-1)\cdots(x^{15}-1)?$$

- (A) 0  (B) 120 (C) 722 (D) 60 (E) 15!

Solution. When we expand the multiplication, from each factor $x^n - 1$ we can choose either x^n or -1 . So each subset $A = \{a_1, \dots, a_k\}$ of the integers $\{1, 2, \dots, 15\}$ whose elements add up to 60, that is $a_1 + a_2 + \dots + a_k = 60$, will contribute as $(-1)^{15-k} x^{60}$. Therefore, the coefficient of x^{60} is equal to the number of such subsets with odd elements minus the number of such subsets with even elements.

On the other hand, $1 + 2 + 3 + \dots + 15 = 120$. So, if the elements of a subset A add up to 60, the elements of the complement of A will add up to 60 as well. If A has k elements, its complement will have $15 - k$ elements. Note that k and $15 - k$ have different parities, so the number of such subsets with odd elements is equal to the number of such subsets with even elements. Thus, the coefficient of x^{60} is zero.

Problem 23. Consider the set S of integral points in the square $\mathcal{O}(0,0)$, $A(0,25)$, $B(25,25)$ and $C(25,0)$, i.e.

$$S = \{(x, y) \mid x, y \in \mathbb{Z}; 0 \leq x, y \leq 25\}.$$

We say that the points P and Q in S are *friends* if the amplitude of the angle $\widehat{POQ}$ is 45° . By convention $\mathcal{O}$ has no friends. Let F be the subset of points in S which have a friend. How many elements does F have?

- (A) 529 (B) 539 (C) 549  (D) 559 (E) 569

Solution. As it turns out, it is easier to count points not in F , i.e., points without friends first.

By symmetry, the point (a, b) has a friend if and only if the point (b, a) has a friend. Moreover, every point of the form (a, a) is friends with the point $(1, 0)$. Thus, we look at the points $P(a, b)$ with $a > b$. Now, the point $P(a, b)$ lies on the line $y = (b/a)x$. We then have that $\tan(\angle POC) = b/a$. Suppose that a point Q is a

friend of P . Then we have that $\angle QOC = \angle POC + 45^\circ$, and so

$$\tan(\angle QOC) = \frac{1 + b/a}{1 - b/a} = \frac{a + b}{a - b}$$

meaning that Q lies on the line $y = (a + b)/(a - b)x$.

We note that the point $(a - b, a + b)$ lies on this line, and that from assumption, $0 < a - b \leq 25$. Thus, if $a + b \leq 25$, we have that $(a - b, a + b)$ is a possible friend for P . In particular, if $a \leq 13$, then $b \leq 12$ and indeed $a + b \leq 25$.

Also, we have that $a + b \leq 50$, so if $\gcd(a - b, a + b) = d > 1$, then $0 < (a - b)/d < (a + b)/d \leq 25$, and the point $((a - b)/d, (a + b)/d)$ is a friend of P . Now, if $a \equiv b \pmod{2}$, then certainly $2 \mid \gcd(a + b, a - b)$ and so $\gcd(a - b, a + b) > 1$, and thus P has a friend. Otherwise, we have that $\gcd(a + b, a - b)$ divides both $(a + b) + (a - b) = 2a$ and $(a + b) - (a - b) = 2b$, meaning that $\gcd(a - b, a + b)$ divides $\gcd(2a, 2b) = \gcd(a, b)$ due to a and b having distinct parity and thus not both being even. On the other hand, we also have that $\gcd(a, b)$ divides both $a - b$ and $a + b$, and thus $\gcd(a - b, a + b)$, and so in fact $\gcd(a - b, a + b) = \gcd(a, b)$.

Thus, the points (a, b) with $a > b$ that do not have a friend are exactly the points satisfying the following conditions:

1. $a > 13$;
2. $25 - a < b < a$;
3. $a \not\equiv b \pmod{2}$ and $\gcd(a, b) = 1$.

We proceed to count the points going over all $14 \leq a \leq 25$. This is a bit long but tractable. There are 63 of these points:

$x = a$	$y = b$
14	13
15	14
16	15, 13, 11
17	16, 14, 12, 10
18	17, 13, 11
19	18, 16, 14, 12, 10, 8
20	19, 17, 13, 11, 9, 7
21	20, 16, 10, 8
22	21, 19, 17, 15, 13, 9, 7, 5
23	22, 20, 18, 16, 14, 12, 10, 8, 6, 4
24	23, 19, 17, 13, 11, 7, 5
25	24, 22, 18, 16, 14, 12, 8, 6, 4, 2

We then have, by symmetry $63 \cdot 2 = 126$ points without friends, i.e., not in F , so F contains $26^2 - 126 - 1 = 549$ elements.

Problem 24. For his birthday, Alice's teacher copies all numbers from $1, 2, \dots, 2026$ on a board in an arbitrary order. He tells the students that they are allowed to erase any two numbers provided that they then write their positive difference. After 2025 steps only one number is left on the board. What is the maximum possible value for this number?

- (A) 1012 (B) 1013 (C) 2001 (D) 2026 (E) 2025 ♡

Solution. Note that the difference of any two numbers a and b is less than or equal to $\max\{a, b\}$, so at each step the difference is at most 2026. Thus, the final number is at most 2026. Furthermore, $|a - b|$ and $a + b$ have the same parity, so after each step the parity of the total sum of the numbers on the board does not change. Since

$$1 + 2 + \dots + 2026 = \frac{2026 \times 2027}{2} = 1013 \times 2027$$

is odd, the final answer will be odd as well. The largest odd number smaller than 2026 is 2025.

The students can reach 2025 by doing the following: Replace each one of these pairs with their difference $(2, 3), (4, 5), (6, 7), \dots, (2024, 2025)$ which is 1. After that the remaining numbers on the board are 2026 along with 1013 copies of 1. Pairing 1012 copies of 1 and replacing each pair by 0 we get 601 copies of 0 and one 1 and one 2026. Then, we can pair zeros with 1 until we are left with 1 and 2026 on the board. The final number will be 2025.

Problem 25. The numbers n and $n + 2$ are positive integers whose product, when divided by 2026, is a nonzero perfect square. If n is as small as possible, what is the sum of the decimal digits of n ?

- (A) 2025 (B) 41 (C) 27 (D) 10 (E) 9 ♡

Solution. We are looking for positive integers n with the property that $n(n + 2) = 2026y^2$ for some positive integer y . After a bit of algebra, this last equation becomes

$$(n + 1)^2 - 2026y^2 = 1. \tag{†}$$

Here is one way to derive a solution to (†). We work in the number system $\mathbb{Z}[\sqrt{2026}]$ consisting of all real numbers that admit a representation in the form $x + y\sqrt{2026}$, with x, y integers. It is simple to check that

- (i) $\mathbb{Z}[\sqrt{2026}]$ is closed under multiplication,
- (ii) every element of $\mathbb{Z}[\sqrt{2026}]$ has a *unique* representation in the form $x + y\sqrt{2026}$, with integers x, y ,

(iii) if we define the **norm** of $x + y\sqrt{2026}$ as $x^2 - 2026y^2$, then the norm of a product of two elements of $\mathbb{Z}[\sqrt{2026}]$ is the product of their norms.

Since $2026 = 45^2 + 1$, there is an obvious element of norm -1 in $\mathbb{Z}[\sqrt{2026}]$, namely $45 + \sqrt{2026}$. By fact (iii),

$$(45 + \sqrt{2026})^2 = 4051 + 90\sqrt{2026}$$

will have norm $(-1)(-1) = 1$. That is,

$$4051^2 - 2026 \cdot 90^2 = 1.$$

Thus, $n = 4050$ is a solution to $(\dagger)$. We will argue below that 4050 is the smallest solution: hence, the answer to the question as stated is $4 + 0 + 5 + 0 = 9$.

Suppose n, y are integers satisfying $(\dagger)$ with $n, y > 0$. Using again that $2026 = 45^2 + 1$, we see that

$$((n+1) - 45y)((n+1) + 45y) = 1 + y^2.$$

Since $n+1+45y > 0$ and $1+y^2 > 0$, also $n+1-45y > 0$. Write $n+1-45y = k$, where $k > 0$. Substituting into the last displayed equation gives $k(90y+k) = 1+y^2$. Using that $k \geq 1$,

$$1 + y^2 = k(90y + k) \geq 90y + 1.$$

Hence, $y \geq 90$, and $n+1 = 45y + k \geq 45 \cdot 90 + 1 = 4051$. So $n \geq 4050$.



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