



UGA High School Varsity Math Tournament
October 25, 2025

WITH SOLUTIONS

TEAM ROUND

TIME: 1 HOUR

LENGTH: 3 PROBLEMS

MAX SCORE 210 POINTS

70 POINTS FOR A CORRECT ANSWER.

Problem 1. What is the smallest number such that, if you write it in base 10, moving the rightmost digit to the leftmost position is the same as multiplying it by 4?

Answer. 102,564

Solution. Let's answer a more general family of questions here: what is the smallest number such that, when you move the rightmost digit to the leftmost position, you get the original number multiplied by N ? Here, we tackle $N = 5, 4, 7$. In particular the case $N = 4$ is the original problem.

Case $N = 5$

The case $N = 5$ is well-known, but it's good to show its solution here. Let x be this number, k the number of digits it has, and a its ones digit. Then we get the equation

$$a \cdot 10^{k-1} + \frac{x - a}{10} = 5x,$$

which simplifies, after some algebra, to

$$49x = a(10^k - 1).$$

Since $a \leq 9$, there is at most one factor of 7 we can get from a , so we need $10^k - 1$ to be a multiple of 7. Fermat's little theorem then gives us that to satisfy this condition, we want k to be a factor of 6, meaning the least k is either 1, 2, 3, or 6. As it turns out, $10 - 1$, $10^2 - 1$, and $10^3 - 1$ are not multiples of 7, but $10^6 - 1$ is.

To keep computations small, we reduce everything mod 7 and ask: when is 10^k , equivalently 3^k , congruent to 1 mod 7? The sequence of residues is easy to compute:

$$3, 2, 6, 4, 5, 1.$$

So you really need to go all the way to $k = 6$.

However, $10^6 - 1$ is not divisible by 49, because $(10^6 - 1)/7 = 142,857$, which is one off from a multiple of 7. So either we make k larger, or we take our other factor of 7 from a . Making k larger would multiply x by a factor of at least 10 (in fact, if you know the Lifting the Exponent Lemma, you know it gets much larger—you'd need to add a factor of 7 to the exponent of 10), so you might as well just put that factor of 7 into a , meaning $a = 7$. Setting $a = 7$, $k = 6$ gives us

$$x = \frac{7(10^6 - 1)}{49} = \frac{10^6 - 1}{7} = 142,857.$$

Indeed, $714,285 = 142,857 \times 5$.

Case $N = 4$

For the original problem, which is the case $N = 4$. Repeating the above, we get

$$39x = a(10^k - 1).$$

As it turns out, there is no need for a to be divisible by 3, because $10^k - 1$ is already always divisible by 3. So we only require $10^k - 1$ to be a multiple of 13. As before, k should be at least 6, and indeed $k = 6$ works. We get

$$x = \frac{a(10^6 - 1)}{39} = 25,641a.$$

Since x should have at least 6 digits, $a = 4$ works, giving $x = 102,564$. Indeed¹, $410,256 = 102,564 \times 4$.

Case $N = 7$

Finally, the case $N = 7$. We get

$$69x = a(10^k - 1).$$

Since $10^k - 1$ is always divisible by 3, we ask only that it be divisible by 23 as well. As before, we need only check $k = 1, 2, 11, 22$. Clearly both 1 and 2 fail. To check that 11 fails, the slick way is to use Euler's criterion and Legendre symbols (quadratic reciprocity); however, barring that, you can just keep multiplying by 10 mod 23:

$$10^2 \equiv 8, \quad 10^4 \equiv 8^2 \equiv 18, \quad 10^8 \equiv 18^2 \equiv (-5)^2 \equiv 2 \pmod{23}.$$

Then $10^{11} = 10^{8+2+1} \equiv 2 \cdot 8 \cdot 10 = 160 \equiv -1 \pmod{23}$. So we need $k = 22$.

Even after this, we get

$$x = \frac{a(10^{22} - 1)}{69},$$

and the smallest value of a giving a 22-digit number is $a = 7$, yielding

$$x = 1014492753623188405797.$$

Indeed,

$$7 \times 1014492753623188405797 = 7101449275362318840579.$$

If we forbid any zeros, the problem becomes even harder. A more tractable version might just ask for the number of digits.

¹If we demand that x have no zero digits, then $a = 6$ is still the correct answer, because both $a = 4$ and $a = 5$ fail this condition.

Finally, there's a more hands-on way to approach this question without the number-theoretic machinery—ironically, easiest for the $N = 7$ case. Clearly x must start with a 1. So suppose we have

$1 \dots a.$

We know from above that $a = 7$ works, but suppose we didn't. Try it and see what happens: moving the 7 to the left gives $71\dots$. Since $71/7 = 10$ remainder 1, the digit to the right of 1 must be 0, so our number is $10\dots7$, and 7 times that number is $710\dots$. Dividing again, $710/7 = 101$ remainder 3, so the next digit must be 1 because our number would then be $7101\dots$. We continue this process until the remainder becomes 0. If it ever failed, we'd know—but it doesn't.

Problem 2. Let N be the smallest positive multiple of 2025 whose decimal digits add up to 2025. Find the sum of the *distinct* decimal digits of N .

Answer. 29

[illegible]

$$N = 8 \cdot 10^{225} - 10^{223} - 25.$$

Hence, the distinct decimal digits in N add up to $7 + 9 + 8 + 5 = 29$.

Since each digit is at most 9, certainly N has at least $2025/9 = 225$ decimal digits. Furthermore, N cannot have exactly 225 digits: that would require N to consist of 225 9s, i.e., $N = 10^{225} - 1$. But that N is not divisible by 5, and so certainly not divisible by 2025. Thus, N has at least 226 decimal digits.

We now consider candidate N values with precisely 226 digits. Since N is a multiple of 25, the final two digits of N are either 00, 25, 50, or 75. Let S denote the sum of these last two digits, and let ℓ denote the leading digit of N . Then the sum of the digits of N is at most $S + \ell + 223 \cdot 9$, and so

$$S + \ell + 223 \cdot 9 > 2025.$$

Hence,

$$S + \ell > 18.$$

Since $\ell \leq 9$, it must be that $S \geq 9$. This forces the last two digits of N to be 75, so that $S = 12$ and $\ell > 6$.

Suppose $\ell = 6$. Since $S = 12$ and the sum of the digits of N is 2025, the 223 digits between the leading 6 and the ending 75 must all be 9. That is, $N = 699 \cdots 975$, or equivalently

$$N \equiv 7 \cdot 10^{225} - 25.$$

However, this N is not divisible by 81, and hence is not divisible by $2025 = 81 \cdot 25$. To see this, we use that 10 has multiplicative order 9 modulo 81 (as can be verified by a straightforward calculation). Since $9 \mid 225$, it follows that $10^{225} \equiv 1 \pmod{81}$. Thus, modulo 81,

$$\begin{aligned} 7 \cdot 10^{225} - 25 &\equiv 7 \cdot 1 - 25 \\ &\equiv -18 \\ &\not\equiv 0. \end{aligned}$$

Hence, our leading digit $\ell \geq 7$. If $\ell = 7$, then the 223 digits between the leading 7 and the ending 75 are all 9, with a single exception which is equal to 8. In other words,

$$N = 8 \cdot 10^{225} - 10^r - 25$$

for some r with $2 \leq r \leq 224$. Each N of this form is divisible by 25, and so N will be divisible by $2025 = 25 \cdot 81$ precisely when N is divisible by 81. Working modulo 81,

$$\begin{aligned} N &\equiv 8 \cdot 10^{225} - 10^r - 25 \\ &\equiv 8 - 10^r - 25 \\ &\equiv -17 - 10^r \\ &\equiv 64 - 10^r. \end{aligned}$$

Thus, N is divisible by 2025 precisely when $10^r \equiv 64 \pmod{81}$. Since $8 \cdot 10 \equiv -1 \pmod{81}$, we have $8^2 \cdot 10^2 \equiv 1 \pmod{81}$. But we have observed already that 10 has order 9 mod 81, and hence $10^7 \cdot 10^2 \equiv 1 \pmod{81}$. Thus, $10^7 \equiv 8^2 \equiv 64 \pmod{81}$, and $10^r \equiv 64 \pmod{81}$ precisely when $r \equiv 7 \pmod{9}$. To make the number N above as small as possible, we make r as large as possible. The largest $r \leq 224$ with $r \equiv 7 \pmod{9}$ is $r = 223$. Hence, $N = 8 \cdot 10^{225} - 10^{223} - 25$.

Problem 3. If you pick a point on an (infinite) cylinder, it always lies on a line contained in that cylinder. This is not true if you take a point on the surface S described by the equation

$$xyz + x + y + z = 0$$

as it only contains finitely many lines. However some points lie on more than one line. How many such points are there?

Answer. 13

Solution. Notice that if we let $x = 0$, the equation above reduces to $y + z = 0$, which gives us our first line,

$$L_x : (0, t, -t).$$

In the same vein, we get the lines $L_y : (t, 0, -t)$ and $L_z : (t, -t, 0)$.

Assume now that $x = 1$, the equation above becomes $yz + y + z + 1 = (y+1)(z+1) = 0$ whence we get two new lines

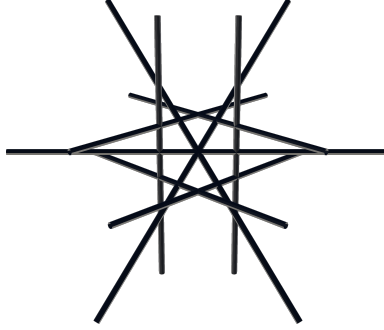
$$L_{xy} : (1, -1, t)$$

and

$$L_{xz} : (1, t, -1).$$

Mutatis mutandis, we obtain four other lines: $L_{yx} : (-1, 1, t)$, $L_{yz} : (t, 1, -1)$, $L_{zx} : (-1, t, 1)$ and $L_{zy} : (t, -1, 1)$.

These lines are represented below.



These are the only lines on S ; indeed, consider a generic line $(a + \alpha t, b + \beta t, c + \gamma t)$ included on this surface, all points on the line must satisfy the equation of S :

$$(a + \alpha t)(b + \beta t)(c + \gamma t) + (a + \alpha t) + (b + \beta t) + (c + \gamma t) = 0$$

which after rearranging the terms, becomes

$$abc + a + b + c + (ab\gamma + a\beta c + b\alpha c + \alpha + \beta + \gamma)t + (a\beta\gamma + b\alpha\gamma + \alpha\beta c)t^2 + \alpha\beta\gamma t^3 = 0.$$

This expression must be true for all t hence all coefficients of this cubic polynomial must be identically 0. In particular, from the cubic coefficient, it implies that one of α , β or γ must be null. Without loss of generality, let's assume that $\alpha = 0$ and the last equation simplifies to

$$0 = abc + a + b + c + (ab\gamma + a\beta c + \beta + \gamma)t + (a\beta\gamma)t^2.$$

Looking at the term of degree 2, we now have two options: either $a = 0$ or, β or γ is 0. If $a = 0$, the equation further reduces to

$$0 = b + c + (\beta + \gamma)t = 0$$

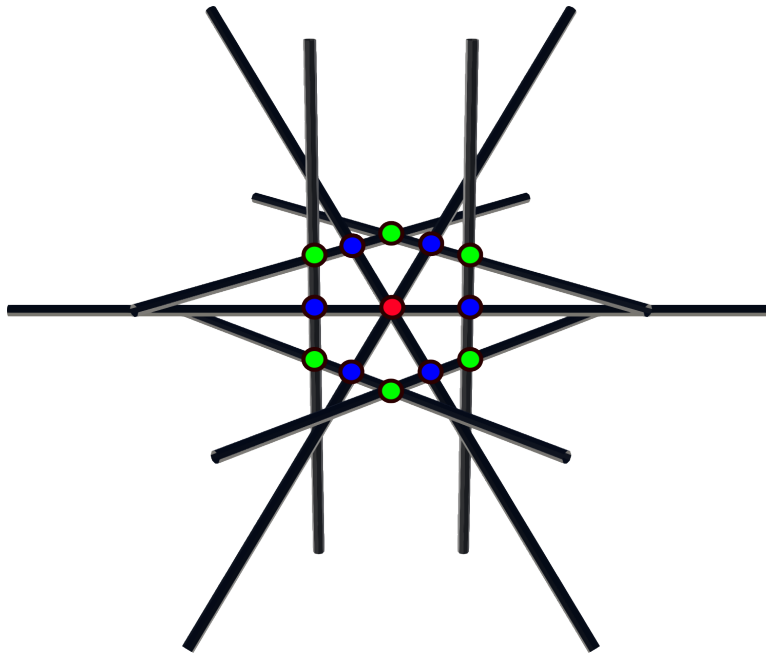
i.e. $y + z = 0$. If say, $\beta = 0$, the equation reduces to

$$0 = abc + a + b + c + (ab + 1)\gamma t$$

Now, γ can't equal 0 or our line would degenerate to a point so we must have $ab+1 = 0$. This constraint and $abc + a + b + c = 0$ reduce to $a + b = 0$ and $-a^2 + 1 = 0$, i.e. $a = -b = \pm 1$, i.e. the lines $(1, -1, t)$ and $(-1, 1, t)$. By symmetry, we get the other four lines.

We now have to find the points of intersection of these nine lines. By observation, we get the following pairs of intersecting lines: L_a, L_{bc} where a, b and c are distinct; L_{ab}, L_{ac} where a, b and c are distinct; L_{ac}, L_{bc} where a, b and c are distinct. Finally, all L_a 's, for $a = x, y, z$ pass through the origin. All in all, these are $6 + 6 + 1 = 13$ points of intersection.

These lines and points of intersection are represented below.



Fun fact: this surface is often called the Cayley cubic. Cubic surfaces have at most 27 lines on them. This is only the tip of a fascinating iceberg.



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