Numerical Analysis Qual Exam (Fall 2015)

All problems are 10 points each.

Problem 1. Both polynomials $P^{(1)}(x) = x^4 - 2x^3 + 2x^2 - 2x + 1$ and $P^{(2)}(x) = x^4 - x^3 + x - 1$ have a root $x^* = 1$. However, the errors between two consecutive Newton's iterations $e_k^{(i)} = |x_{k+1}^{(i)} - x_k^{(i)}|, k \geq 0$ with $x_0^{(i)} = 1.2$ are shown in the following table for $P^{(i)}, i = 1, 2$:

k	0	1	2	3	4
$i = 1$	$0.910 \cdot 10^{-1}$	$0.517 \cdot 10^{-1}$	$0.278 \cdot 10^{-1}$	$0.145 \cdot 10^{-1}$	$0.741 \cdot 10^{-2}$
$i = 2$	0.152	$0.448 \cdot 10^{-1}$	$0.329 \cdot 10^{-2}$	$0.163 \cdot 10^{-4}$	$0.398 \cdot 10^{-9}$

The errors for $P^{(1)}$ decrease much slower.

a) (2 points) Explain the reason for this phenomenon.

b) (2 points) What can be done to accelerate the convergence for the first polynomial $P^{(1)}$?

c) (6 points) Justify your suggestion with a mathematical proof.

Problem 2.

a) (5 points) Define the Gaussian quadrature $G_n(f)$ carefully for a continuous function f.

b) (5 points) State the convergence theorem of the Gaussian quadratures $G_n(f)$ to the integral of f and give a proof.

Problem 3. Suppose that $A \in R^{n \times n}$ is nonsingular. Let $\vec{u}, \vec{v} \in R^n$ be two vectors. Under what condition is the following matrix

$$\begin{pmatrix} 0, & \vec{u}^T \\ \vec{v}, & A \end{pmatrix}$$

invertible (4 points)? Find the explicit form for the inverse matrix (6 points).

Problem 4. Let

$$U_n(x) := \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}}, \quad n = 0, 1, 2, \ldots.$$

a) (5 points) Prove that these functions are algebraic polynomials of degree n.

b)(5 points) For integers $n \neq m$, prove

$$\int_{-1}^{1} \sqrt{1-x^2}U_n(x)U_m(x)dx = 0.$$

Problem 5. Fix $n \geq 1$. Let B_n be a polynomial B-spline of degree n with integer nodes and support on $[0, n+1]$. Prove that $\{B_n(x-i)\}_{i=-n}^{N-1}$ are linearly independent on the interval $[0, N]$, $N \geq 1$.

Problem 6. An $(n+1)$-dimensional linear subspace H of $C[a, b]$ is called a Haar subspace if each non-zero function in H has at most n roots.

Show that the linear span H of functions $\{1, x, x^2, \cdots, x^{n-1}, f(x)\}$ is a Haar subspace of $C([a, b])$ if the n^{th} derivative $f^{(n)}(x)$ of f is strictly positive on $[a, b]$.

Problem 7. Suppose that the matrix norm $\|\cdot\|$ is subordinate. Let S be a non-singular square matrix. Prove or disprove that $\|A\|_* := \|SAS^{-1}\|$ is also a subordinate norm.

Problem 8. Suppose that a square matrix A is strictly diagonally dominant. Show that the Gauss-Seidel iteration for the linear system $A\mathbf{x} = \mathbf{b}$ converges.

Problem 9. Consider a single step method $y_{k+1} = y_k + h\psi(x_k, y_k, h)$ for numerical solution of initial value problem of ODE $y' = f(x, y)$. Suppose that $\psi(x, y, h)$ is Lipschitz continuous with respect to y with Lipschitz constant L. Suppose that the local truncation error of order m, i.e., $T_k(h) = \dfrac{y(x_{k+1}) - y(x_k)}{h} - \psi(x_k, y(x_k), h) = \mathcal{O}(h^m)$. Show that numerical solution y_k approximates $y(x_k)$ in the following sense

$$|y(x_k) - y_k| \leq e^{(b-a)L}|y(x_0) - y_0| + \frac{e^{(b-a)L} - 1}{L}Ch^m,$$

for $k = 1, \cdots, n$.

Problem 10. Consider a linear least squares problem:

$$\min \|A\mathbf{x} - \mathbf{b}\|^2,$$

where A is a matrix of size $m \times n$ with $m > n$ and $\mathbf{b}$ of size $m \times 1$. (a) Use the SVD to describe how to solve the least square problem (5 points), and (b) explain why your method works (5 points).