Numerical Analysis Qualifying Examination

Spring, 2007

Name____

Instruction: *Among the following 9 problems, please do 8 of them. Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours.*

[1] Suppose that a square matrix A is strictly diagonally dominant. Show that when applying the Gaussian elimination procedure with partial pivoting to A, no partial pivoting is needed.

[2] Suppose that a square matrix A is strictly diagonally dominant. Show that when applying Gauss-Seidel iteration to solve $Ax = b$, the iteration converges.

[3] Let $\widetilde{A} = A + E$ be a perturbation matrix of A. Show that

$$\|A^+ - \widetilde{A}^+\|_2 \le \|E\|_2 \left(\|A^+\|_2 \|\widetilde{A}\|_2 + \|A^+\|_2^2 + \|\widetilde{A}^+\|_2^2 \right)$$

where A^+ and $\widetilde{A}^+$ denote the pseudo inverses of A and $\widetilde{A}$, respectively.

[4] Let A be a tridiagonal matrix. Write $A = QR$ with R being upper triangular matrix and Q an orthonormal matrix. Show that RQ is also a tridiagonal matrix.

[5] Use the Steepest Descent Method to solve

$$g(\mathbf{x}^*) = \min_{\mathbf{x} \in \mathbf{R}^n} g(\mathbf{x})$$

where $g(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t A \mathbf{x} - \mathbf{x}^t \mathbf{b}$, $n = 3$, and

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Start with $\mathbf{x}^0 = (0, 0, 0)^t$ and perform three iterations of the Steepest Descent Method.

[6] Suppose that $f \in C^2[a, b]$. Prove the following error for the midpoint rule:

$$\int_a^b f(x)dx - (b-a)f(\frac{a+b}{2}) = \frac{(b-a)^3}{24} f''(\xi)$$

for some $\xi \in [a, b]$.

[7] Let $G_n(f) := \sum_{k=1}^n f(x_{n,i})c_{n,i}$ be the n^{th} Gaussian quadrature formula over the interval $[a, b]$, where $x_{n,i}, i = 1, \cdots, n$ are zeros of the orthogonal polynomial ϕ_n of degree n and

$$c_{n,i} = \int_a^b \prod_{\substack{j \ne i \\ j=1,\cdots,n}} \frac{(x - x_{n,j})}{(x_{n,i} - x_{n,j})} dx.$$

Show that $c_{n,i} > 0$ for all i.

[8] Let $\triangle := \{x_0, \cdots, x_{n+1}\}$ be a partition of $[a, b]$, i.e, $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$. Let $S_3^1(\triangle)$ be the space of all C^1 cubic spline functions. That is, for any $s \in S_3^1(\triangle)$, $s \in C^1[a, b]$ and $s|_{[x_i, x_{i+1}]}$ is a cubic polynomial, $i = 0, \cdots, n$. For any $f \in C^1[a, b]$, let S_f be the C^1 cubic interpolatory spline of f, i.e., $S_f \in S_3^1(\triangle)$ and

$$S_f(x_i) = f(x_i), S'_f(x_i) = f'(x_i), i = 0, 1, \cdots, n + 1.$$

Suppose that $f \in C^2[a, b]$. Show that

$$\int_a^b |\frac{d^2}{dx^2}(f(x) - S_f(x))|^2 dx \leq \int_a^b |\frac{d^2}{dx^2}(f(x) - s(x)|^2 dx$$

for any $s \in S_3^1(\triangle)$.

[9] Consider solving the following initial value problem numerically.

$$\begin{cases} y'(x) = f(x, y(x)), & a \leq x \leq b. \\ y(a) = \alpha, \end{cases}$$

Verify the following Taylor method of order 2: $h = (b - a)/n, y_0 = \alpha$ and for $k = 0, \cdots, n - 1$,

$$y_{k+1} = y_k + hf(x_k, y_k) + \frac{h^2}{2} f_x(x_k, y_k) + \frac{h^2}{2} f_y(x_k, y_k) f(x_k, y_k).$$