Numerical Analysis Qualifying Exam

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January 9, 2016

Instruction: *The following are 8 problems in total. Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours.*

1. Use Taylor's theorem for a function of two variables to carefully derive and state Newton's method for the numerical solution of the system of two nonlinear equations:
$$\begin{cases} f(x,y)=0, \\ g(x,y)=0. \end{cases}$$
Give a sufficient condition which ensures that the Newton method converges.

2. Let $\mathbf{x}$ and $\tilde{\mathbf{x}}$ be the solution of two linear systems: $A\mathbf{x}=\mathbf{b}$ and $\tilde{A}\tilde{\mathbf{x}}=\tilde{\mathbf{b}}$. Show
(a) If $A=\tilde{A}$, then
$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \le \text{cond}(A)\frac{\|\mathbf{b}-\tilde{\mathbf{b}}\|}{\|\mathbf{b}\|}.$$
where $\text{cond}(A)=\|A\|\|A^{-1}\|$ stands for the condition number of A.
(b) Consider the case that $A\neq\tilde{A}$. Show that if $\|A^{-1}\|\cdot\|A-\tilde{A}\|<1$, then
$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \le \frac{\text{cond}(A)\left[\frac{\|A-\tilde{A}\|}{\|A\|}+\frac{\|\mathbf{b}-\tilde{\mathbf{b}}\|}{\|\mathbf{b}\|}\right]}{1-\ \text{cond}(A)\frac{\|A-\tilde{A}\|}{\|A\|}}.$$

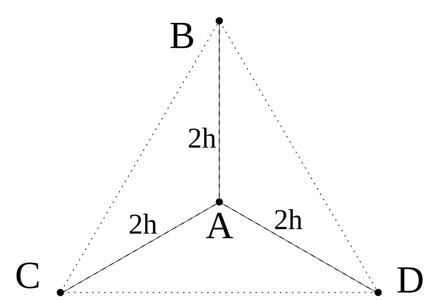
3. Recall that for any nonzero vector v of size $n\times 1$, $I-2\dfrac{vv^T}{\|v\|^2}$ is called a Householder matrix, where I is the $n\times n$ identity matrix. Show that there exists a sequence of Householder matrices $H_1,\dots,H_n$ which converts any matrix A into a lower triangular matrix L, that is, $H_n\cdots H_1 A=L$. (Hint: you may use a matrix of 4×4 to explain how to do.)

4. Let f be a continuous function on $[a,b]$. The following statements are true or false. If it is true, give some reasons, e.g., quote a well-known theorem. If it is false, give an example.
(1) There exists a sequence of polynomials p_n such that p_n converges to f uniformly.
(2) There exists a sequence of interpolatory polynomials $p_n(f)$ which interpolates f at $n+1$ distinct points $x_i^{(n)}\in[a,b], i=0,1,\cdots,n$ such that $p_n(f)$ converges to f uniformly.
(3) Let $x_i^n=a+i(b-a)/n, i=0,\cdots,n$. The interpolatory polynomial $p_n(f)$ at these x_i^n's converges to f pointwisely.

5. Let $p_n(x) = \sum_{i=0}^{n} c_i B_i^n(x)$ be a polynomial in B-form with respect to $[a,b]$. Here, $B_i^n(x) = \binom{n}{i}\left(\frac{x-a}{b-a}\right)^i\left(\frac{b-x}{b-a}\right)^{n-i}$ is defined on the interval $[a,b]$. Similarly, let $q_n(x) = \sum_{i=0}^{n} d_i \tilde{B}_i^n(x)$ with $\tilde{B}_i^n(x) = \binom{n}{i}\left(\frac{x-b}{c-b}\right)^i\left(\frac{c-x}{c-b}\right)^{n-i}$ defined on $[b,c]$. Derive the conditions on their coefficients of p_n and q_n that ensure

$$\frac{d^r}{dx^r}p_n(b) = \frac{d^r}{dx^r}q_n(b), \quad \forall r = 0,1,2.$$

6. Find an approximation for $\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ using the following four function evaluations: $f(x,y)$, $f(x,y+2h)$, $f(x-\sqrt{3}h, y-h)$ and $f(x+\sqrt{3}h, y-h)$, which form the three corners and the center of an equilateral triangle as shown below. Note that the points A, B, C and D correspond to the locations of these given function values, respectively.



7. Show that when the composite trapezoid rule is applied to $\int_a^b e^x dx$ using equally spaced points, the relative error is exactly

$$1 - \frac{h}{2} - \frac{h}{e^h - 1}$$

8. Derive the general 2^{nd}order Runge-Kutta method for numerical solution of ODE, where α is a variable:

$$\begin{aligned} K_1 &= h\,f(t, x(t)) \\ K_2 &= h\,f(t+\alpha h, x(t) + \alpha K_1) \\ x(t+h) &= x(t) + AK_1 + BK_2 \end{aligned}$$

In other words, express A and B in terms of α.