MATHEMATICS PRELIMINARY EXAM: FALL, 2009

(1) Consider the following statement about real numbers, which we will call S:

"If $x \leq 0$ and $x < y$, then there is a z such that $x \cdot z > y$."

(This happens to be false, but for the purposes of this problem, that is irrelevant.)

(a) Using the symbols $\forall$ (for all), $\exists$ (there exists), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (implies), $\neg$ (not), write S in symbolic form. Be sure to include implicit quantifiers.

(b) Write the negation of S, in verbal form, without using word "not".

(c) Write the converse of S, in verbal form.

(2) Work the following Calculus problems.

(a) Compute the derivative of $f(x) = \sec^3(\tan(x^2))$.

(b) Find the Taylor expansion for $f(x) = \ln(x)$ about $a = 2$, and determine its radius of convergence.

(c) Set up (but do not evaluate) the triple integral needed to compute the integral of $f(x, y, z) = x \sin(y + z)$ over the region R in $\mathbb{R}^3$ defined by $x \geq 0$, $y + z \geq 2$ and $x^2 + y^2 + z^2 \leq 4$.

(3) Fix $x \in \mathbb{R}$. Using mathematical induction, show that if $x \neq 1$, then for $n = 1, 2, 3, \cdots$ we have

$$x + 2x^2 + 3x^3 + \cdots + nx^n \;=\; \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2} \;.$$

(4) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Give the ε-δ definition for the existence of the limit of $f(x)$ as x approaches x_0. Then, use the definition to show that $\lim_{x \to 1/4} \sqrt{x} = 1/2$.

(5) (a) Show that for the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) \;=\; \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \;, \\ 0 & \text{if } (x, y) = (0, 0) \;, \end{cases}$$

we have $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$, but that f is not even continuous at $(0, 0)$.

(b) Give a sufficient condition for a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ to be differentiable at a point P, in terms of the partial derivatives of g.

(6) Let V be the vector space over $\mathbb{R}$ spanned by the functions $f_1(x) = 1$, $f_2(x) = \sin^2(x)$, $f_3(x) = \cos^2(x)$, $f_4(x) = \sin(2x)$, and $f_5(x) = \cos(2x)$. Find the dimension $\dim_{\mathbb{R}}(V)$, and exhibit a basis for V over $\mathbb{R}$.

(7) If A is the 2×2 complex matrix below, find an invertible complex matrix P for which $P^{-1}AP$ is diagonal.

$$A \;=\; \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

(8) Let $f : [0, 1] \rightarrow \mathbb{R}$ and $f_n : [0, 1] \rightarrow \mathbb{R}$, for $n = 1, 2, 3, \cdots$, be continuous functions. Suppose the sequence $\{f_n\}_{n \geq 1}$ converges uniformly to f. Give the definition of uniform convergence, and using basic properties of integrals and absolute values, show that

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx \;=\; \int_0^1 f(x)\, dx \;.$$