Mathematics Preliminary Exam, Fall 2011

1. Assume the following statement is true: "If Michael got at least 80 on his final exam, then he passed his history class".

a) Write the contrapositive of this statement.
b) Write the converse of this statement.
c) What, if anything, can be concluded if Michael's final exam score was 75?
d) What, if anything, can be concluded if Michael passed the course?
e) What, if anything, can be concluded if Michael failed the course?

2. Find an invertible matrix A and a diagonal matrix B such that $\begin{pmatrix} 8 & 9 \\ -6 & -7 \end{pmatrix} = ABA^{-1}$.

3. Give an example (with proof) of a power series that converges exactly on the interval $(1, 3]$.

4. Give an ϵ, δ proof that $\lim_{x\to 2} \dfrac{1}{3-x} = 1$.

5. Let C be the upper semi-circle of radius 3 in the plane with base the line from $(-3, 0), (3, 0)$. Orient C counterclockwise. Compute $\oint_C (2x+y)dx+ydy$ both directly and by using Green's theorem.

6. Use induction to show that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

7. Let $f : X \to Y$ be a map of sets and let $S \subset Y$. Give, with proof, necessary and sufficient conditions for $f(f^{-1}(S)) = S$.

8. Draw representative level sets (i.e. $f^{-1}(c)$ for $c \in \mathbb{R}$) for $f(x, y) = x^2 - y^2$. Next, compute and draw the gradient, ∇f, at a representative collection of points in $\mathbb{R}^2$. Explain how these two objects must be related in general.

9. Say two elements of $\mathbb{R}^3$ are equivalent, and write $(x, y, z) \sim (x', y', z')$, if $x + y + z = x' + y' + z'$. Find a basis for the vector space $V = \mathbb{R}^3/\sim$ and prove it is a basis.