Mathematics Preliminary Exam, Fall 2013

1. A function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
   a) Give a definition of what it means to not be uniformly continuous.
   b) Give an example, with brief explanation, of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous, but not uniformly continuous.

2. Suppose A, B, and C are $n \times n$ matrices such that A is invertible and $ABA^{-1} = C$. Give quick proofs (citing relevant facts about trace, determinant, etc.) of whichever of the following statements are true:
   a) The determinants of B and C are equal.
   b) The traces of B and C are equal.
   c) The eigenvectors of B and C are the same.
   d) The eigenvalues of B and C are the same.

3. Give an example (with proof) of a power series that converges exactly on the interval $[2, 4)$.

4. Use the definition of a limit to show that $\lim_{x\to 2} \dfrac{1}{x^2 - 1} = 1/3$.

5. a) Let C be a path from $(0, 1)$ to $(1, 2)$. Explain whether or not $\int_C ye^x dx + xe^y dy$ depends on the choice of C.

   b) Give an example of a function defined on the plane whose gradient always points towards the origin.

6. Show that for all $k \in \mathbb{N}$ that $s_k = 1/1 + 1/2^2 + \cdots 1/k^2$ is less than or equal to $2 - 1/k$. Prove or disprove: s_k is a Cauchy sequence.

7. Let $f : X \to Y$ be a map of sets and let $A, B \subset X$, and $C, D \subset Y$. In each of the following choose the best relation between between the sets, $\subset, \supset$, or $=$. No proofs are necessary.
   a) $f(A \cap B) \qquad f(A) \cap f(B)$

   b) $f^{-1}(C - D) \qquad f^{-1}(C) - f^{-1}(D)$

   c) $f(f^{-1}(C)) \qquad C$.

8. Let z be the complex number $1 + \sqrt{3}i$. Express z^3 and all of the cube roots of z in the form $a + bi$.

9. Give an example, with proof, of a linear map from the vector space $\mathbb{R}^3$ to some other vector space which has kernel equal to the subspace spanned by $(1, 2, 3)$.