Department of Mathematics
PRELIMINARY EXAMINATION
August 12, 2014

3 hours. Work all of the following problems. Justify your answers.
$\mathbb{R}$ denotes the field of real numbers;
$\mathbb{N}$ denotes the set $\{1, 2, 3, \dots\}$ of natural numbers.

1. Let f be a function mapping $\mathbb{R}$ of to $\mathbb{R}$. Give the definitions of each of the following concepts in words, without using logical symbols or the word "not".
   a) f is surjective
   b) f is not injective
   c) f is uniformly continuous
   d) f is not uniformly continuous

2. Let (a_n) be a sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{a_n}{3} + 5$ for $n \geq 1$. Prove inductively that $a_n \leq a_{n+1} \leq 10$ for each positive integer n. Then explain why the sequence (a_n) converges and find its limit.

3. Suppose f and g are functions mapping $\mathbb{R}$ into itself with $\lim_{x \to 0} f(x) = 0$.
   a) Prove from the $\epsilon - \delta$ definition that if g is bounded, then $\lim_{x \to 0} f(x)g(x) = 0$ as well.
   b) Give an example to show that the boundedness hypothesis cannot be omitted from Part a).

4. Let v_1, v_2, v_3 form a basis for a vector space V. Prove that $v_1 + v_2,\ v_2 - v_3,\ v_2 + 2v_3$ form a basis for V.

5. Let R be the region in the first quadrant bounded by the curves $x^2 + y^2 = 2x$ and $y = 0$. Let C be the boundary of the region R, oriented counterclockwise. Evaluate the line integral

$$\int_C xe^x dx \ + \ (ye^y + x^2)dy.$$

6. Take $A := \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$. Find an orthogonal matrix P for which $P^{-1}AP$ is diagonal. Then find the minimum value of the dot products $Ax \cdot x$ as x ranges through the unit vectors in $\mathbb{R}^2$.

7. Take $A = \{0, 1, 2, 3, 4, 5\}$. Define a relation R on A by $x\,R\,y$ if and only if $x^2 - 4x = y^2 - 4y$.
   a) Prove that R is an equivalence relation.
   b) Exhibit the partition of A whose members are the equivalence classes of R.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous.
   a) Define what it means for f to be *(totally) differentiable* at the origin.
   b) Suppose $\frac{\partial f}{\partial x}$ is continuous at $(0, 0)$, while $\frac{\partial f}{\partial y}$ exists at $(0, 0)$. Prove that f is differentiable at $(0, 0)$.

   Note: You can "buy" the answer to Part a) for 4 points. Also you can earn partial credit on Part b) by making the stronger assumption both partial derivatives are continuous at $(0, 0)$.