Preliminary Exam, Spring 2009

(3 hours, 8 problems counted equally)

1. Write the precise definition for what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be uniformly continuous. Then without using the word "not", state precisely and in elegant prose what it means for f to fail to be uniformly continuous.

2. Prove from the ϵ–δ definition that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \frac{1}{x}$ is continuous at 3.

3. Prove that for each natural number n there is a function $f : \mathbb{R} \to \mathbb{R}$ whose nth derivative is given by $f^{(n)}(x) = \frac{1}{\sqrt{3+x^4}}$.

4. Prove that if $\sum a_n$ is convergent and $\sum b_n$ is absolutely convergent, then $\sum a_n b_n$ is also absolutely convergent. What can be said if the series $\sum a_n$ and $\sum b_n$ are only conditionally convergent ?

5. Suppose V is a finite-dimensional vector space and $T : V \to W$ is a linear transformation. Give self-contained proofs that

$$\mathrm{Range}(T) := \{y \in W : y = Tx \text{ for some } x \in V\}$$

is a subspace of W and $\dim(\mathrm{Range}(T)) \leq \dim V$.

6. Suppose A is a self-adjoint matrix in $M_n(\mathbb{R})$ whose only eigenvalues are 0 and 1. Prove that $A^2 = A$. Carefully state any "big theorem(s)" you use in your argument.

7. Determine, with proof, which, if either, of the following functions mapping $\mathbb{R}^2$ into $\mathbb{R}$ are continuous.

a) $$f(x,y) = \begin{cases} \frac{x^3 y}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & x = y = 0 \end{cases}$$

b) $$g(x,y) = \begin{cases} \frac{x^3 y^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & x = y = 0 \end{cases}$$

8. Examples/Computations. No proofs are required.

a) Give an example of a matrix $A \in M_2(\mathbb{R})$ that is not diagonalizable.

b) Calculate the line integral $\int_C xy^2\, dy$ where C is the right triangle with vertices $(0,0), (1,0)$, and $(0,1)$, oriented counterclockwise.

c) Compute $f^{(10)}(0)$ for the function $f(x) = \cos(x^2)$.