Algebra PhD qual remarks for students. (Written by Roy Smith, August 2006)

It is difficult to give a single reference where one can find proofs, examples, applications, and intuitive explanation of all topics. This makes it incumbent on the student to seek this multi level understanding by combining several sources. In general Dummit and Foote (DF) excels in illustrative examples, but the proofs there may sometimes not be complete or may not be the simplest. Of course this is somewhat a matter of taste. Smith's notes focus more on proofs and have fewer examples and exercises. DF spends roughly 600 pages on the graduate qual material while Smith covers it in about 350 pages. Hungerford is another good reference, as well as Artin, Lang, and Van der Waerden.

Graduate material:
Group theory:
For group theory, the first 6 chapters (220 pages) of DF are excellent, with the following comments. They define isomorphisms as bijective homomorphisms, which to modern minded people is a theorem, an isomorphism always being technically a homomorphism with a homomorphism inverse. The proofs there of Sylow theorems and simplicity of A_n for n > 4 look good. The proof of the most fundamental isomorphism theorem, and the proof of the Jordan Holder theorem, one easy, one hard, are both left to the exercises. The proof JH is seldom tested on the qual, but proofs are always of interest. Of course doing proofs as exercises is beneficial if one can do them.

The discussion in DF of the meaning and applications of the fundamental theorem of finite abelian groups, and of cyclic groups, looks excellent, but the proofs of the theorem given in chapters 6 and 12 are not constructive. The constructive proof, needed to find actual decompositions, is left as exercises #16-#19 in chapter (12.1). Note that in applying the theorem to normal forms of matrices, DF appeal to the constructive version in the exercises.

All the definitions and proofs of these theorems on groups are given in Smith's web based lecture notes for math 843 part 1. There is more than one proof of Sylow's theorems, a complete proof the only simple groups of order between 60 and 168 have prime order, plus an introduction to the categorical and functorial point of view, in only 80 pages, but far fewer examples and exercises than in the first 220 pages of DF.

Rings:
All the necessary material on rings is covered in DF, chapters 7,8,9 (roughly 115 pages).
Smith's notes treat rings in a somewhat scattered way, in sections 13, 14, of 843 -2 , and the first 5 sections of 844-1, (43 total pages), mainly as needed to treat Galois theory. The main results are proved, but the most elementary facts are assumed as similar to those for groups. Gauss's important theorem on unique

factorization of polynomials is proved twice however in Smith for clarity, once for Z[X], and again for R[X], R any ufd.

Comments:

1. DF again define a ring isomorphism as a bijective homomorphism, rather than giving the categorical definition as a homomorphism with an inverse homomorphism. (Of course this matters less in algebra than in topology, algebraic geometry, and analysis, where it actually gives the wrong notion, but it seems a bad habit to acquire.)

2. The fundamental isomorphism theorems for rings are not fully proved in DF, but are easy and useful exercises. They are not even stated in Smith's notes, who takes the attitude they should be obvious by this point. DF is more systematic in the first few sections of chapter 7, to state again most of the elementary facts about homomorphisms, ideals and quotient rings, analogous to those for groups. Smith has little discussion of basic ring properties. One crucial condition DF inexplicably do not even state however (but Smith does), is when a ring map f, out of a ring R, induces a ring map out of a quotient R/I, namely whenever f(I) = {0}.

3. Zorn's lemma is clearly used in DF to find maximal ideals (p.254), and clearly stated in the appendix. The proof that Zorn follows from Ax. Ch. is not given, but this is never tested on quals, and seldom needed by non logicians. It can be found in Lang if desired. Smith also discusses extensively in 844 - 1, how to use Zorn to find maximal ideals, to construct algebraic field closures, and why Zorn is unnecessary in countable or noetherian rings.

4. The definition of Euclidean domain is not universally agreed on. The one in Smith's notes (p.27, section 5, 844-1), includes the condition that the norm of a product N(ab) is at least as large as that of a factor N(a), which DF do not assume. DF can still deduce that a Euclidean domain is a p.i.d. and hence a u.f.d., but the argument for the existence of an irreducible factorization is harder, making the proof that Z and k[X] are ufd's (p.289, p.300 DF) unnecessarily difficult. This can make those cases longer to prove on a qual exam.

An interesting topic covered in DF p.277, but not in Smith is a necessary criterion for a ring to be a Euclidean domain, allowing DF to give a simple example of a non Euclidean p.i.d.

5. The section 8.2 on p.i.d.'s in DF looks good but the history may be faulty. They give a trivial result about the equivalence of pid's and rings with "Dedekind - Hasse" norms, attributing one direction (p.281) to Greene in 1997. This result appears already in Zariski - Samuel vol.1, p.244, in 1958, and is so easy it was surely known to Dedekind.

6. The nice remark in DF, p.305, that a polynomial ring over a ufd, in infinitely many variables is still a ufd, is easy, but not always noted (cf. exercise in Smith 844-1, section 5).

7. The statement of Eisenstein's criterion in DF p.309, is unnecessarily weak, since it is stated only for monic polynomials, although the proof of the non monic case is the same. Smith's notes and most other books have the full version. There is also a much stronger version not often seen, using a Newton polygon argument on the graph of the points, one for each term of the polynomial, whose (s,t) coordinates are the exponents of X and of some prime factor of the coefficients, due to G. Dumas, in Van der Waerden, vol. 1, 2nd edition, p.76.

8. The example of the cyclic product structure of (Z/nZ)* is an exercise in DF, and is explained in more detail in Smith (844 -2, section 18).

9. DF includes a nice introduction to Grobner bases and division algorithms for polynomials in several variables, 9.6.

Modules over pid's and Canonical forms of matrices.
The treatment in DF in sections 10.1, 10.2, 10.3, and 12.1, 12.2, 12.3, appears to be excellent (77 pages) but the constructive proof of the decomposition is given only as exercises 16-19, in 12.1 of DF. The treatment in Smith 845-1 and 845-2, (also 77 pages), includes a detailed discussion of the constructive proof.

Field theory/ Galois theory.
DF in chapters 13,14, seem to give an excellent treatment of field and Galois theory, (about 145 pages). Smith treats it in 843-2, sections 11,12, and 16-21 (39 pages), 844-1, sections 7-9 (20 pages), 844-2, sections 10-16, (37 pages).