2002

**EIGHTH ANNUAL**
**MATHEMATICS COMPETITION**
WRITTEN TEST, 25 PROBLEMS / 90 MINUTES

UNIVERSITY OF GEORGIA
OCTOBER 26, 2002

**WITH SOLUTIONS**

**No calculators are allowed on this test.** 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

## 1. Easy Problems

**Problem 1.** What is the number of integers between 1 and 1000 that are not divisible by 2 or 5?

A) 100 B) 200 C) 250 D) 300 E*) 400

**Solution.** 1000 minus 500 (those divisible by 2) minus 200 (those divisible by 5) plus 100 (the ones divisible by 10 we subtracted twice); equals 400.

**Problem 2.** Two trains 6 miles apart start running towards each other with speeds 7 and 5 miles per hour respectively. A fly flies from the first train to the second, turns around, flies back, turns around, flies to the second, turns around etc., until it gets squashed. How many miles did the fly cover if its speed is 8 miles per hour?

A) 2 B) 3 C*) 4 D) 5 E) 6

**Solution.** The trains will meet in $6/(7+5) = 0.5$ hours, so the fly will cover $8 \cdot 0.5 = 4$ miles.

**Problem 3.** Find $x$ if

$$2^{16^x} = 16^{2^x}.$$

A*) 2/3 B) 2 C) 4 D) −1 E) None of the above

**Solution.**

$2^{16^x} = 16^{2^x} = 2^{4 \cdot 2^x} \Rightarrow 16^x = 4 \cdot 2^x \Rightarrow 2^{4x} = 2^{2+x} \Rightarrow 4x = 2+x \Rightarrow x = 2/3$

**Problem 4.** Let $p_i$ denote the $i^{\text{th}}$ prime number. How many solutions in positive integers $n$ and $k$ does the following equation have:

$$\prod_{i=1}^{n} p_i = k^2 - 1?$$

A*) 0 B) 1 C) 2 D) 3 E) infinitely many

**Solution.** The left hand side is divisible by 2 but not by 4. The right hand side is always either 0 or 3 modulo 4. Therefore, there are no solutions.

**Problem 5.** Two students attempted to solve a quadratic equation $x^2 + ax + b = 0$. Although both students did the work correctly, the first miscopied the middle term and obtained the solution set $\{2, -2\}$. The second miscopied the constant term and obtained the solution set $\{-1, -2\}$. What is the correct solution set?

A*) $\{-4, 1\}$ B) $\{-2, -1\}$ C) $\{-2, 0\}$ D) $\{-1, -1\}$
E) None of the above

**Solution.** From the first result we have $b = x_1 x_2 = -4$, and from the second: $a = -x_1 - x_2 = 3$. Hence, the original equation was $x^2 + 3x - 4 = 0$, and the solutions set is $\{-4, 1\}$.

**Problem 6.** What is the number of ways to color the six faces of a cube six different colors? (Two ways are considered to be the same if one can be obtained from another by rotating the cube.)

A) 15 B*) 30 C) 64 D) 120 E) 720

**Solution.** There are 6! ways to color, and there are $8 \cdot 3$ rotations. Therefore, there are

$$\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{8 \cdot 3} = 5 \cdot 6 = 30$$

distinct colorings.

**Problem 7.** An ant is crawling from a corner of a cube to another corner which is farthest from the first, always crawling along edges and never crossing its path or returning. Find the number of ways it can do that, provided that the first and the last corners are fixed.

A) 12 B*) 18 C) 27 D) 72 E) None of the above

**Solution.**

$$3 \cdot 2 \cdot (1 + 1 \cdot (1 + 1)) = 6 \cdot 3 = 18$$

**Problem 8.** How many words can you spell with the letters POTATO? (The "words" do not have to make sense).

A) 30 B) 90 C*) 180 D) 360 E) 720

**Solution.**

$$\frac{6!}{2!\ 2!} = 6 \cdot 5 \cdot 3 \cdot 2 = 180$$

Indeed, changing the order of letters TT and OO does not change the word.

**Problem 9.** Find

$$\sum_{n=1}^{7} n \cdot n!$$

A) 5039 B) 35279 C) 35281 D*) 40319
E) None of the above

**Solution.**

$$n \cdot n! = (n+1)! - n!,$$

so

$$\sum_{n=1}^{7} n \cdot n! = (2! - 1!) + (3! - 2!) + \ldots + (8! - 7!) = 8! - 1! = 40319$$

**Problem 10.** A ball is shot from a corner of a square billiard table with side 1 meter at an angle arctan(8/15). It bounces around until it falls into some corner pocket. What total distance in meters does the ball travel?

A*) 17 B) $\sqrt{283}$ C) 23 D) 120 E) $\infty$

**Solution.** Instead of bouncing, draw a grid and observe that a straight line starting from the point (0,0) at this angle will next time intersect an integral point at (8,15). Therefore, the length is

$$\sqrt{8^2 + 15^2} = 17$$

**Problem 11.** Find the sum of digits in all 3-digit integers whose digits are 1, 2, 3, 4, or 5, with no repetitions allowed.

A) 270 B) 450 C*) 540 D) 630 E) None of the above

**Solution.** There are $5 \cdot 4 \cdot 3$ such integers, and the average of the sum of the digits is $3 \cdot 3$. Hence, we get

$$5 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = 540$$

**Problem 12.** A drawer contains 6 blue socks and 4 white socks. Two socks are selected randomly (without replacement). What is the probability that the socks are the same color?

A) 1/2 B) 6/15 C) 8/15 D) 2/3 E*) None of the above

**Solution.** There is a probability of $\frac{3}{5} \times \frac{5}{9} = \frac{1}{3}$ that both socks drawn are blue; there is a probability of $\frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$ that both socks drawn are white. Therefore, the probability that socks of the same color are drawn is $\frac{1}{3} + \frac{2}{15} = \frac{7}{15}$.

## 2. Medium Problems

**Problem 13.** What is the largest number of Friday the $13^{\text{th}}$s possible in a calendar year?

A) 1  B) 2  C*) 3  D) 4  E) 5

**Solution.** Assign an arbitrary number from 0 to 6 to January 13th. We can then compute the corresponding numbers modulo 7 for all other months by adding 31=2 mod 7 for January, 28=0 mod 7 or 29=1 mod 7 for February, etc. Depending on the leap or non-leap year, we get the following sequences for January, February, . . . December:

144025036146

034025036146

The answer is the maximal number of equal digits in a line: the same numbers fall on the same number of the week. Hence, the answer is 3.

**Problem 14.** What is the smallest number of Friday the $13^{\text{th}}$s possible in a calendar year?

A) 0  B*) 1  C) 2  D) 3  E) 4

**Solution.** As in the solution to the previous problem, we need to see what is the minimal number of a certain digit from 0 to 6 that can appear in a line.

**Problem 15.** In how many pieces can one cut a cake with 8 straight cuts?

A) 23  B) 29  C*) 37  D) 256  E) None of the above

**Solution.** With every cut, the maximum we can add is the number of lines that the new line intersects plus one. Hence, the answer is

$$1 + (1 + 2 + 3 + \cdots + 8) = 1 + \frac{8 \cdot 9}{2} = 37$$

**Problem 16.** What is the number of integers between 1 and 360 which are not divisible by 2, 3, or 5?

A) 48 B*) 96 C) 108 D) 192 E) None of the above

**Solution.** $400 = 2^3 \cdot 3^2 \cdot 5$, and the answer is

$$(2^3 - 2^2)(3^2 - 3)(5 - 1) = 4 \cdot 6 \cdot 4 = 96$$

(this is called the Euler function $\phi(360)$).

**Problem 17.** Denote by $x$ the sum of the following infinite series:

$$x = \frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \dots$$

Express the sum

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$$

in terms of $x$.

A) $7x/8$ B) $9x/8$ C*) $8x/7$ D) $2x$ E) None of the above

**Solution.** Denote

$$y = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$$

Then

$$\begin{aligned} x &= \left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots\right) - \left(\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \frac{1}{8^3} + \dots\right) \\ &= \left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots\right) - \frac{1}{2^3}\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots\right) \\ &= y - \frac{1}{2^3}y = \frac{7}{8}y \end{aligned}$$

Therefore, $y = 8x/7$.

**Problem 18.** Find the sum

$$\sum_{k=1}^{5} k!(k^2 + 3k + 1)$$

A) 5037 B) 5039 C) 5759 D) 5763 E*) None of the above

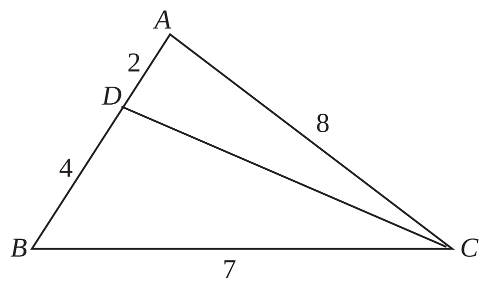
**Solution.** This is an example of a "telescoping sum". One checks that

$$(k+2)! - k! = k!\,((k+2)(k+1)-1) = k!(k^2+3k+1)$$

Therefore, our sum equals

$$(3!-1!)+\dots(7!-5!) = (7!-1!)+(6!-2!) = (5040-1)+(720-2) = 5757$$

**Problem 19.** In a triangle $ABC$ one has $AC = 8$ and $BC = 7$. The point $D$ lies on $\overline{AB}$ between points A and B, and $AD = 2$, $DB = 4$. Find the length $CD$.




A) $\sqrt{46}$ B) $\sqrt{46.5}$ C) $4\sqrt{3}$ D*) $\sqrt{51}$ E) None of the above

**Solution.** Applying the law of cosines to triangles ADC and BDC, we get:

$$x^2 + 4 + 4x\cos\theta = 64$$
$$x^2 + 16 - 8x\cos\theta = 49 \Rightarrow$$
$$3x^2 + 24 = 177 \Rightarrow x^2 = 51 \Rightarrow x = \sqrt{51}$$

**Problem 20.** Find the ninth digit from the right in the number $101^{10}$.

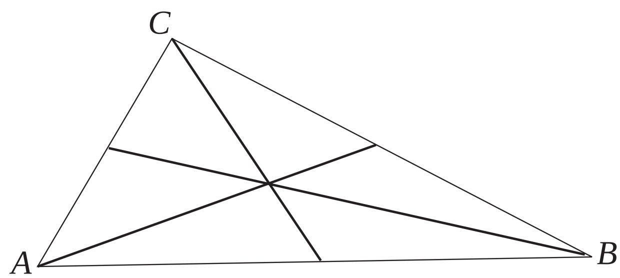
A) 0 B*) 1 C) 2 D) 3 E) 4

**Solution.** We use the binomial expansion for $101^{10}$. Take the 10th row in the Pascal's triangle and, starting from the right, write down the numbers with a shift of two digits when moving to the next one. We get

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 0 | 1 |
| | | | | | | | 1 | 0 | | |
| | | | | | 4 | 5 | | | | |
| | | 1 | 2 | 0 | | | | | | |
| 2 | 1 | 0 | | | | | | | | |
| | | 1 | | | | | | | | |

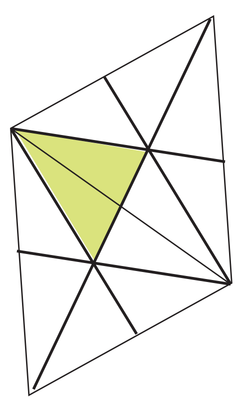
## 3. Hard Problems

**Problem 21.** In triangle $ABC$ the lengths of medians are 3, 4, and 5. Find the length of the shortest side of this triangle.




A*) 10/3 B) 3 C) $\sqrt{13}$ D) $\sqrt{14}$ E) None of the above

**Solution.** Call the sides $a$, $b$, $c$, and the medians $m_a$, $m_b$, $m_c$. As we see from the picture, in a triangle with sides $(2/3)m_a$, $(2/3)m_b$, $(2/3)m_c$ the medians have lengths $a/2, b/2, c/2$. Therefore, we need to compute the medians in the 3-4-5 triangle and multiply them by 4/3. This is easy to do, and the smallest of these numbers will be $(5/2) \cdot (4/3) = 10/3$.



**Problem 22.** You decide to build a dream house. Your main requirement is: the house has to be twice as close to UGA as it is to the beach (for the sake of simplicity, the beach is assumed to be a straight line, and UGA to be a point; moreover, UGA is not on the beach). What is the locus of points where you can build your house?

A) a straight line B) a parabola C) a hyperbola D) a circle
E*) an ellipse but not a circle

**Solution.** Choose the rectangular system of coordinates so that the line has equation $y = 0$ and the point has coordinates (0,1). Then the equation for the locus is

$$y = 2\sqrt{x^2 + (y-1)^2} \Leftrightarrow y^2 = 4(x^2 + (y-1)^2) \Leftrightarrow$$
$$4x^2 + 3y^2 - 8y + 4 = 0 \Leftrightarrow 4x^2 + 3(y - \frac{4}{3})^2 = \frac{4}{3}$$

This is an equation of an ellipse.

**Problem 23.** Eight teams are playing in an elimination style tournament (such as NCAA basketball March madness tournament). Therefore, they are split into two groups of 4 teams, which are then split into two groups of 2. In how many ways can this be done? Two ways are considered to be the same if all groups of 2 and groups of 4 are the same. (For example, tournaments 12|34||56|78 and 21|34||56|78 are considered to be the same: in both cases the groups of 2 are 12, 34, 56, 78, and the groups of 4 are 1234, 5678).

A) 256 B*) 315 C) 5040 D) 40320 E) None of the above

**Solution.** The answer is 8! divided by the maximal power of 2, so that the answer is an odd number, i.e.

$$\frac{1}{1} \cdot \frac{2}{2} \cdot \frac{3}{1} \cdot \frac{4}{4} \cdot \frac{5}{1} \cdot \frac{6}{2} \cdot \frac{7}{1} \cdot \frac{8}{8} = 3 \cdot 5 \cdot 3 \cdot 7 = 315$$

Indeed, there are 8! ways to number the eight teams 1 through 8. Next, we have to eliminate double counting. In each groups of 2 the order can be changed, so divide by $2^4$, and in each of group of 4 the two pairs can be exchanged, so divide by $2^2$. Finally, the groups of 4 can be exchanged, so divide by 2.

The same answer and proof work for the number of tournaments with $2^n$ teams for any $n$: $2^n!$ divided by the maximal power of 2 it contains.

**Problem 24.** Choose the largest of the following numbers

A) $(\frac{1}{e})^{(\frac{1}{\pi})}$ B) $(\frac{1}{e})^{(-\frac{1}{\pi})}$ C) $(\frac{1}{\pi})^{(-\frac{1}{e})}$ D) $\pi^e$ E*) $e^\pi$

**Solution.** The choice is obviously between $e^\pi$ and $\pi^e$. After taking the logarithms on both sides and simple manipulation, we must compare $(\ln e)/e$ and $(\ln \pi)/\pi$. Look now at the function $f(x) = (\ln x)/x$. One has

$$f'(x) = \frac{1 - \ln x}{x^2}$$

Then $f'(e) = 0$, $f'(x) > 0$ for $x < e$, and $f'(x) < 0$ for $x > e$. It follows that the point $x = e$ is the only maximum. Therefore, $e^\pi$ is larger.

**Problem 25.** For one of the cubic polynomials below the roots form an arithmetic progression. Which one?

A) $x^3 + 3x^2 - 2x - 1$ B) $x^3 + 3x^2 - 2x - 2$ C) $x^3 + 3x^2 - 2x - 3$
D*) $x^3 + 3x^2 - 2x - 4$ E) $x^3 + 3x^2 - 2x - 5$

**Solution.** The roots of a cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ satisfy $x_1 + x_2 + x_3 = -a$. Therefore, they form an arithmetic progression if and only if $-a/3$ is a root, i.e. $p(-a/3) = 0$. This gives the condition

$$(-a/3)^3 + a(-\frac{a}{3})^2 + b(-\frac{a}{3}) + c = 0 \quad \text{or} \quad 2(\frac{a}{3})^3 = (\frac{a}{3})b - c$$

Only the polynomial $x^3 + 3x^2 - 2x - 4$ satisfies this condition.

**Authors.** Valery Alexeev and Boris Alexeev, ©2002; with assistance from Ted Shifrin. Not all problems are original. Some sources of inspiration are:

- Polish national competitions for problems 11 and 25.
- Mu Alpha Theta competition 1992 for problem 5.
- Problem 2: too well known to cite a source.