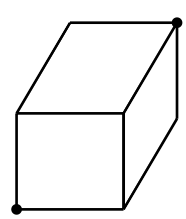
Sponsored by: UGA Math Department, UGA Math Club,
UGA Parents and Families Association

Team Round / 45 minutes

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 50 points, for a total of 200 points.

Problem 1. (Crawling Ant) An ant is crawling on a surface of a rectangular box with sides 9, 10, and 11 cm (see the picture). What is the smallest distance it must crawl to get from one corner to the opposite corner, farthest from the first one?



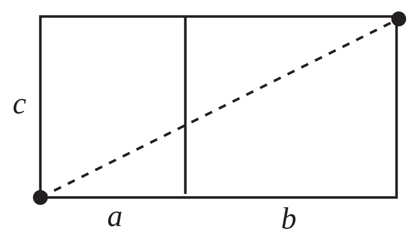
Answer. $\sqrt{482}$

Solution. Call the sides of the box a, b, and c. Clearly, the ant should use no more than two sides of the box. We can then fold these two sides and put them on the plane to obtain a rectangle with sides $a+b$ and c (see the picture). Then shortest path from one corner of this rectangle to another is the diagonal, and its distance is d, where

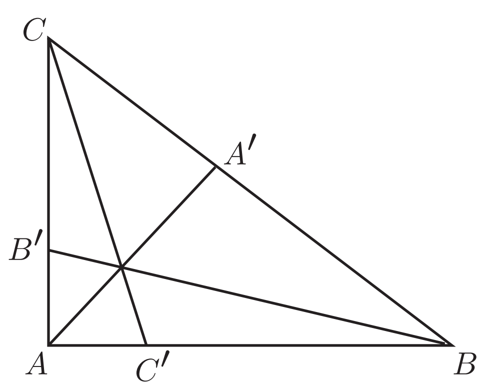
$$d^2 = (a+b)^2 + c^2 = (a^2+b^2+c^2) + 2ab$$

by the Pythagorean theorem. We have six possibilities for a, b and c. To minimize d, we must minimize $2ab$, therefore, a and b are 9 and 10, and $c = 11$. Therefore,

$$d = \sqrt{19^2 + 11^2} = \sqrt{482}$$



Problem 2. (A triangle problem) In a triangle ABC with right angle A, one has $AB = 4$ and $AC = 3$. On the sides AB, BC and CA the points C', A', and B' are chosen in such a way so that the lines AA', BB', CC' intersect at one point, and $AB' = 1$, $AC' = 1$ (see the picture). Find BA'.



Answer. 3

Solution. First of all, $BC = \sqrt{3^2 + 4^2} = 5$. Secondly, by Ceva's Theorem,

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1$$

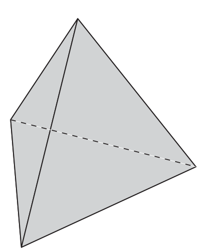
Denoting BA' by x, we get

$$\frac{1}{3} \cdot \frac{x}{5 - x} \cdot \frac{2}{1} = 1$$

Solving this linear equation, we get $x = 3$.

Problem 3. (Irregular dice) One can check (but you don't have to) the following amazing fact: if you throw two irregular dice – one with numbers 1, 3, 4, 5, 6, 8 painted on it, and another with 1, 2, 2, 3, 3, 4 – then the sums of the two numbers (i.e. 2, 3, ..., 12) will appear with the same probabilities as if you were throwing two regular dice, with the numbers 1 through 6 painted on them.

Now, here is your problem: find two irregular tetrahedral dice (with 4 faces instead of 6), so that the sums (i.e. 2, 3, ..., 8) appear with the same probabilities as if you were throwing two regular dice. The numbers painted must be positive integers and, as in the example above, they may repeat. Your answer must be two sets of 4 integers each.



Answer. 1, 2, 2, 3 and 1, 3, 3, 5 (the order does not matter).

Solution. *First way (educated trial and error).* As for regular dice, sum 2 must appear once, hence, on both dice number 1 has to appear once, without repetition. Arguing the same way, the largest numbers N_1 and N_2 on the dice have to appear once, and we must have $N_1 + N_2 = 4 + 4 = 8$. There are two possibilities: $N_1 = N_2 = 4$ or $N_1 = 3$, $N_2 = 5$. Trying our luck, let us try the second possibility. Then by what we just said, on the first die the other sides must be painted 2 and 2. Hence, the first die has numbers $1, 2, 2, 3$; and the second: $1, x, y, 5$. By going through a few more trial and errors, we get to $x = y = 3$.

Second (better) way. Suppose that on the first die the number 1 appears a_1 times, number 2 appears a_2 times etc. Similarly, we have numbers $b_1, b_2 \ldots$ for the second die. Look at the product of two polynomials

$$(a_1x^1 + a_2x^2 + \ldots)(b_1x^1 + b_2x^2 + \ldots)$$

Then the coefficient of x^n in this polynomial will be the number of possibilities for the sum after throwing the dice to be equal to n. For example, there are $a_1b_3 + a_2b_2 + a_3b_1$ ways to get 4 as the sum, etc.

For the regular dice, we get in this way

$$(x + x^2 + x^3 + x^4)^2 = x^2(1 + x + x^2 + x^3)^2 = x^2(1 + x)^2(1 + x^2)^2$$

To find a different pair of dice, we need to factor this polynomial as $P(x)Q(x)$ in a different way. Here, $P(x)$ and $Q(x)$ must be polynomials with nonnegative coefficients, without constant terms, and such that $P(1) = Q(1) = 4$. One obvious possibility is

$$P(x) = x(1 + x)^2 = x + 2x^2 + x^3, \qquad Q(x) = x(1 + x^2)^2 = x + 2x^3 + x^5,$$

which gives our answer. With a little work, you can check that there is no other way to factor.

Problem 4. (The jackpot) The odds of winning the jackpot in the MegaBux lottery are 1 in 70 million. This week, 70 million tickets were sold for the latest drawing. What is the probability, in percent, that at least one winning ticket was sold? Round your answer to the closest integer.

Answer. 63% (will accept anything between 61% and 65% inclusive).

Solution. The solution uses the famous formula

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

The probability that a given ticket is *not* a winning ticket is

$$1-\frac{1}{70,000,000}$$

Therefore, the probability that there are no winning tickets is

$$\left(1-\frac{1}{70,000,000}\right)^{70{,}000{,}000} \approx e^{-1} = \frac{1}{e} \approx \frac{1}{2.718} \approx .368 \approx 37\%$$

Hence, the probability that there is at least one winning ticket is about 63%.