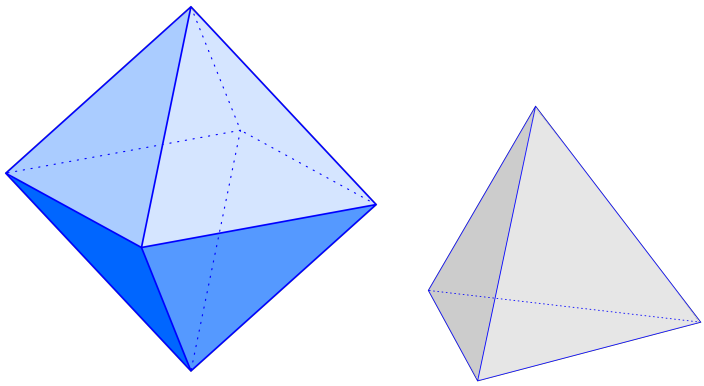
Sponsored by: UGA Math Department and UGA Math Club

TEAM ROUND / 1 HOUR / 210 POINTS
November 16, 2013

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Octahedron vs tetrahedron)**.** Let O be the volume of a regular octahedron with edge length 1, and let T be the volume of a regular tetrahedron with edge length 1. Find the ratio $\frac{O}{T}$.



Problem 2 (Nonstandard primes)**.** By a **binary string**, we mean a finite nonempty sequence of 0s and 1s, with no leading 0s unless the string consists only of 0. Listing strings by length, the first few examples are thus 0, 1, 10, 11, 101, ... We define **non-carry addition** ($+$) and **non-carry multiplication** ($\times$) of binary strings by the usual grade-school algorithms for addition and multiplication **but systematically ignoring carries**. For example, $1 + 1 = 0$ with our definition, and

$$\begin{array}{r} 10101 \\ +\ 1101 \\ \hline 11000 \end{array}$$

while

$$\begin{array}{r} 10101 \\ \times\quad 1101 \\ \hline 10101 \\ 00000 \\ 10101 \\ 10101 \\ \hline 11101001 \end{array}$$

A **prime** is a binary string with more than one digit which cannot be written as a non-carry product except as $1 \times$ itself or itself $\times 1$. For example, 10 and 11 are prime, but 11101001 is not.

How many primes are there with exactly six digits?

Problem 3 (More rectangular boxing)**.** You may recall that on problem #15, you found that the distance from P to Q on the surface of a $1 \times 1 \times 2$ rectangular box is $\sqrt{8}$. (The dashed lines in the figure below show one path that achieves this minimum.) Surprisingly, Q is **not** the point on the surface of the box which is farthest from P. Find the distance from P to the point that is farthest from P.