1785

Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
November 8, 2014

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Lights out)**.** In the game of "Lights out" there is a collection of lights, some of which are on, and some off. If you touch any light, that light and all of the adjacent lights will change; those that are on will turn off, and those that are off will turn on. You win if you can touch a sequence of lights so that all of the lights are off. Depending on which lights are initially on, this may not be possible.

For this problem, there are 8 lights, located at the corners of a cube, so that each light is adjacent to 3 other lights. How many "winning positions" are there? In other words, for how many initial configurations of on/off lights is it possible to turn off all of the lights? Having all of the lights initially off counts as a winning position (you've already won!).

Problem 2 (Counting lattice points)**.** Partition the plane into 1×1 squares using the lines $x = n$ and $y = m$ for all integers m and n. Then draw a circle of radius 100 centered at $(0, 0)$. How many of the 1×1 squares does the circle pass through the interior of? Notice that the circle passes through the interior of the square whose lower left corner is $(0, 99)$, but it does not pass through the interior of the square whose lower left corner is $(0, 100)$.

Problem 3 (Primitive vertices)**.** Let P_n denote the regular n-gon centered at the origin and having one vertex at $(1,0)$. We adopt the convention that P_1 consists of the single point $(1,0)$ and that P_2 consists of the line segment connecting $(-1,0)$ and $(1,0)$. A vertex of P_n is called *primitive* if it is not a vertex of P_m for any $m < n$. For example, each vertex of P_3 is primitive except $(1,0)$. Let C_n denote the center of mass of the primitive vertices of P_n. For how many $n \leq 100$ is C_n located at $(0,0)$?

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Team ID:

Team name:

Answer 1:

Answer 2:

Answer 3: