Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

November 18, 2006

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

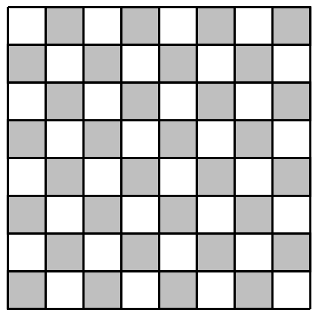
5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit through the rear doors.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. A straight line crosses an 8 by 8 chess board. What is the largest number of squares it can pass through (not just touch the boundary)?



(A) 8 (B) 15 (C) 16 (D) 17 (E) None of the above

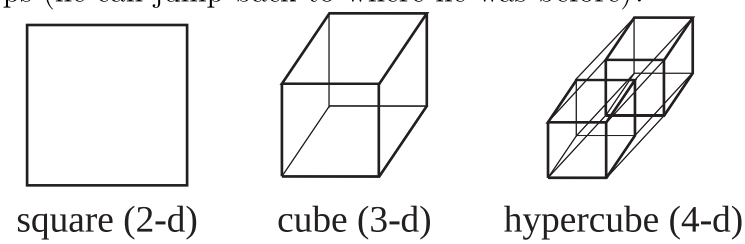
Problem 2. The lion always lies on Mondays, Tuesdays, and Wednesdays, but always tells the truth on the other days. The unicorn always lies on Thursdays, Fridays, and Saturdays, but always tells the truth on the other days. If they both announce to you, "I told lies yesterday," what day is it?

(A) Monday (B) Wednesday (C) Thursday (D) Saturday
(E) None of the above

Problem 3. You go to visit the penguins at the zoo. The penguins live in a very cold circular room with walls made of the glass so that you can see them. You notice one especially cute baby penguin standing against the wall. He waddles 15 feet, bumps into the glass, turns 90 degrees, waddles another 36 feet, bumps into glass again, and falls down. What is the diameter of the penguins' habitat?

(A) 25 (B) 26 (C) 27 (D) 39 (E) None of the above

Problem 4. An ant is hopping from vertex to vertex along the edges of a 4-dimensional hypercube. To how many places can he get in exactly three hops (he can jump back to where he was before)?



(A) 4 (B) 8 (C) 11 (D) 15 (E) None of the above

Problem 5. A king rules over a vast subset of the Cartesian plane, including a river which occupies all points (x, y) such that $0 < y < 1$. The king wishes to build a road from his castle at $(-2, -4)$ to the queen's castle at $(10, 3)$ that will include a bridge (an interval of length 1 crossing the river at a constant x-coordinate x_0). What choice of x_0 minimizes the length of the road?

(A) 4 (B) 6 (C) 8 (D) 9 (E) None of the above

Problem 6. How many positive integers divide 15! (fifteen factorial)?

(A) 2048 (B) 3168 (C) 4032 (D) 4096 (E) None of the above

Problem 7. How many different rectangles whose length and width are both integers have their area equal to 3 times their perimeter?

(A) 1 (B) 2 (C) 4 (D) 5 (E) infinitely many

Problem 8. What is the largest integer n that cannot be represented as $8a + 15b$ with a and b nonnegative integers?

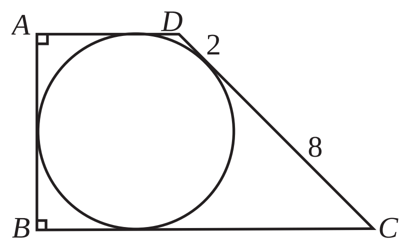
(A) 89 (B) 97 (C) 117 (D) 119 (E) None of the above

Problem 9. Find

$$\frac{1}{1} + \frac{1}{2} + \frac{2}{2} + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \cdots + \frac{9}{10} + \frac{10}{10}$$

(A) $\frac{55}{2}$ (B) 30 (C) $\frac{65}{2}$ (D) 37 (E) None of the above

Problem 10. A circle is inscribed in a trapezoid $ABCD$. If $\angle DAB = \angle ABC = 90°$ and the circle's point of tangency divides line segment $\overline{CD}$ into segments of length 2 and 8, what is the perimeter of the trapezoid?



(A) 32 (B) 36 (C) 40 (D) 48 (E) None of the above

Problem 11. Ten people each have exactly one unique secret. When one of them calls another, the caller tells every secret he knows, but learns nothing from the person he calls. How many phone calls will be needed in order for each person to know all ten secrets?

(A) 18 (B) 19 (C) 20 (D) 21 (E) None of the above

Problem 12. Three circles of radius 1 are tangent to one another. What is the radius of the smallest circle that contains all of them?

(A) 2 (B) $1+\frac{2\sqrt{3}}{3}$ (C) $1+\sqrt{2}$ (D) $\pi/2$ (E) None of the above

Problem 13. The square of an integer has tens digit 7. What is the units digit?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Problem 14. What is the length of the shortest path from the point $(-2, 0)$ to $(2, 0)$ that avoids the interior of the circle of radius 1 centered at the origin?

(A) $2\sqrt{3} + \pi/3$ (B) $2\sqrt{5}$ (C) $2 + \pi$ (D) $2\sqrt{5 - 2\sqrt{2}} + \pi/2$
(E) None of the above

Problem 15. How many paths are there starting from $(0, 0, 0)$ and ending at $(2, 2, 2)$ where each step consists of increasing exactly one of the three coordinates by 1?

(A) 90 (B) 120 (C) 180 (D) 729 (E) None of the above

Problem 16. In how many ways can we arrange 7 (identical) white balls and 5 (identical) black balls in a row so that there is at least one white ball between any two black balls?

(A) 35 (B) 48 (C) 56 (D) 120 (E) None of the above

Problem 17. Find the length of the longest possible geometric progression in $\{100, 101, 102, \ldots, 1000\}$.

(A) 5 (B) 6 (C) 7 (D) 8 (E) None of the above

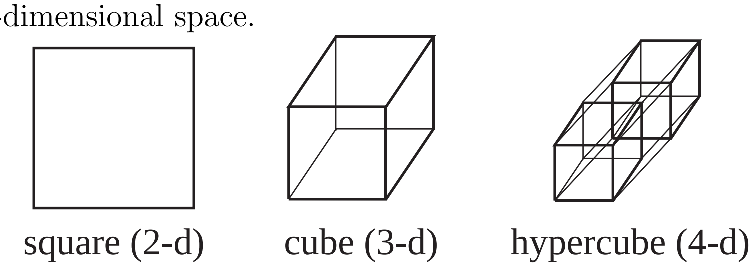
Problem 18. Suppose a, b, c, and n are positive integers. How many solutions of $n! = a! + b! + c!$ are there? (The different ways of permuting a, b, and c count as the same solution.)

(A) 0 (B) 1 (C) 2 (D) 3 (E) None of the above

Problem 19. What is the probability of getting no successive heads when one flips a fair coin 7 times in a row?

(A) 27/128 (B) 1/4 (C) 33/128 (D) 17/64
(E) None of the above

Problem 20. A major diagonal in a 4-dimensional hypercube goes from a vertex to the opposite vertex farthest from it. What is the smallest nonzero angle between two major diagonals? The angle is measured in the 4-dimensional space.

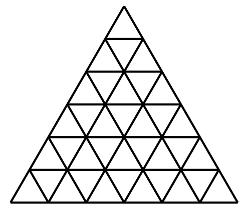


(A) 30° (B) 45° (C) 60° (D) 90° (E) None of the above

Problem 21. Bob has a bag with two coins, one of them fair and one of them double-headed. He randomly chooses one of the coins from the bag and flips it three times. Given that it landed heads each time, what is the probability that it is the fair coin?

(A) $\frac{1}{16}$ (B) $\frac{1}{9}$ (C) $\frac{1}{8}$ (D) $\frac{1}{4}$ (E) None of the above

Problem 22. How many paths are there from A to B which do not pass through any vertex twice and which move only downwards or sideways, never up?



(A) 1024 (B) 2048 (C) 23040 (D) 46080
(E) None of the above

Problem 23. What is the remainder left upon dividing the binomial coefficient $\binom{2006}{1118}$ by 13?

(A) 0, 1, or 2 (B) 3, 4, or 5 (C) 6, 7, or 8 (D) 9 or 10
(E) 11 or 12

Problem 24. Suppose a sequence $a_0, a_1, \ldots$ is chosen randomly, with each a_i independently either $+1$ with probability $3/4$ or -1 with probability $1/4$. What is the probability that for some $n \geq 0$, we have $\sum_{i=0}^{n} a_i < 0$?

(A) 1/3 (B) 1/2 (C) 2/3 (D) 1 (E) None of the above

Problem 25. Boris wanders along a river and gets lost, two miles from home in one direction and three miles from his friend's house in the other, but he doesn't remember which is which. Every hour, Boris randomly and with equal probability chooses a direction and walks a mile in that direction (thus perhaps repeatedly walking along the same stretches of the river). He stops when he reaches one of the two homes. What is the probability that it is his own?

(A) 1/2 (B) 2/3 (C) 3/4 (D) 4/5 (E) None of the above

Authors. Problems by Boris and Valery Alexeev, with contributions by Meredith Perrie, Tyler Kelly, Ted Shifrin and Mo Hendon. Some problems taken from olympiad problems from around the world collected by John Scholes at `http://www.kalva.demon.co.uk`, as well as from a newspaper comic.