Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 13, 2007

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit through the rear doors.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. How many 2-digit numbers are there such that the sum of the digits plus the product of the digits equals the number itself?

(A) 2 (B) 9 (C) 10 (D) 18 (E) None of the above

Problem 2. Tyler has in his pocket assorted coins (including possibly pennies, nickels, dimes, quarters, and fifty-cent pieces). What is the largest possible amount of money he can have without being able to make change for a nickel, a dime, a quarter, a fifty-cent piece or a dollar?

(A) \$.74 (B) \$.69 (C) \$.79 (D) \$.94 (E) None of the above

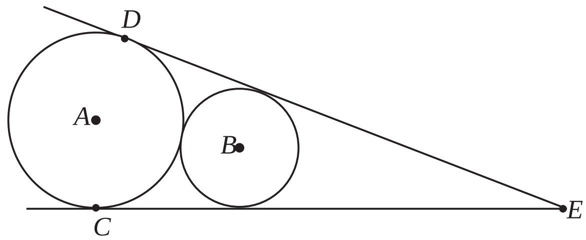
Problem 3. What is length of the side of the largest square with one vertex at the origin that will fit inside the parabola $y = x^2$?

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $\sqrt{3}$ (E) 3

Problem 4. What is the locus of points equidistant from a circle and a point in the plane not lying on the circle?

(A) a parabola (B) an ellipse (C) one branch of a hyperbola
(D) either an ellipse or a branch of a hyperbola (E) either an ellipse or a hyperbola

Problem 5. As shown in the diagram, circles A and B are tangent to each other and to the rays $\overrightarrow{EC}$ and $\overrightarrow{ED}$. If the radius of circle A is 3 and $CE = 4$, then what is the radius of circle B?



(A) 1/2 (B) 5/7 (C) 3/4 (D) 1 (E) None of the above

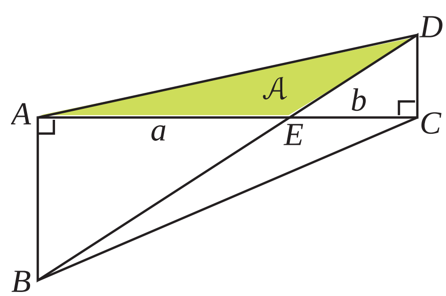
Problem 6. What is the sum of all 4-digit numbers in which the digits 1,2,3,4 appear exactly once?

(A) 33,330 (B) 66,660 (C) 133,320 (D) 399,960 (E) None of the above

Problem 7. How many ways are there to put 8 indistinguishable rooks on an 8-by-8 chess board so that no rook threatens another? (Each rook threatens the rooks which are in the same column or the same row.)

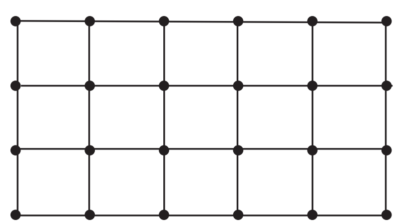
(A) 64 (B) 40,320 (C) 80,640 (D) 1,625,702,400 (E) None of the above

Problem 8. If the area of the shaded triangle $\triangle AED$ is $\mathcal{A}$ and the lengths $AE = a$ and $EC = b$ are given, then the area of quadrilateral $ABCD$ is



(A) $\mathcal{A}\dfrac{(a+b)^2}{ab}$ (B) $\mathcal{A}\dfrac{a^2+b^2}{ab}$ (C) $\mathcal{A}\dfrac{ab}{a^2+b^2}$ (D) $\mathcal{A}\dfrac{ab}{(a+b)^2}$
(E) not enough information

Problem 9. In the rectangular graph pictured below, every edge has length 1. If you draw a Hamiltonian circuit (i.e., a path along the edges that goes through each vertex exactly once, beginning and ending at the same vertex), what is the area of the region it encloses?



(A) 9 (B) 11 (C) 13 (D) 15 (E) The area depends on the Hamiltonian circuit.

Problem 10. The year 2002 was a palindrome: it reads the same forward and backward. The previous palindrome year was 1991. What is the maximal number of non-palindrome years one could have in a row (between 1000 and 9999)?

(A) 10 (B) 11 (C) 79 (D) 81 (E) 109

Problem 11. Consider the real solutions of the equation

$$x^4 - 10x^2 + 1 = 0\,.$$

What is the product of all their cubes?

(A) -10 (B) -1 (C) 1 (D) 10 (E) $10\sqrt{6}$

Problem 12. The words that can be made by rearranging the letters in the word TOPOLOGY are listed alphabetically. What is the 2007^{th} word?

(A) GTOLPOYO (B) OGYOPOLT (C) OGYOPLOT (D) PLOOTOYG (E) None of the above

Problem 13. What is the area of a 13-14-15 triangle?

(A) 80 (B) 82 (C) 84 (D) 86 (E) 88

Problem 14. How many zeroes follow the last nonzero digit of 2007!?

(A) 100 (B) 250 (C) 400 (D) 500 (E) None of the above

Problem 15. Five percent of a group of people are drug-users. Everyone in the group is administered a drug test which is known to be 90% accurate (i.e., 90% of users taking the test are shown to be users, and 90% of nonusers taking the test are shown to be nonusers). If the test indicates that a random person in the group is a user, (approximately) what is the probability that this is a "false positive," i.e., that this result is inaccurate?

(A) 10% (B) 32% (C) 50% (D) 68% (E) 90%

Problem 16. Suppose n is a positive integer. There is a unique fraction with denominator between 500 and 1000 of the form

$$\frac{n+3}{n^2+7n+5}$$

that is *not* in lowest terms. When we put this fraction in lowest terms, its numerator is

(A) 3 (B) 4 (C) 5 (D) 7 (E) 8

Problem 17. What is the maximal number of rooks that can one put in an 8 by 8 by 8 cube so that no rook threatens another? (Each rook threatens the rooks which are in the same row, column, or vertical.)

(A) 8 (B) 64 (C) 512 (D) 40,320 (E) None of the above

Problem 18. Alex and Meredith play a game beginning with n stones. During each turn, either player can remove any number of stones that is 1 less than a prime number; the player who takes the last stone wins. Assuming Alex and Meredith are equally smart players and Meredith goes first, which value of n guarantees victory for Alex?

(A) 2 (B) 5 (C) 9 (D) 11 (E) 14

Problem 19. We start with two pieces of rope 1 meter long and two pegs in the ground 1 meter apart. The pieces of rope are knotted together at one end and the other ends are attached to the respective pegs. To the knot another rope of length 1 meter is attached, and at the other end of this rope there is a cow. What is the area (in square meters) of the region on which the cow can graze?

(A) $\frac{8\pi}{3}$ (B) $4\pi - \frac{\sqrt{3}}{2}$ (C) $\frac{16\pi}{3} - \frac{\sqrt{3}}{2}$ (D) $6\pi - \frac{\sqrt{3}}{2}$ (E) None of the above

Problem 20. In how many distinct ways can one write 1,000,000 as the product of three positive integers? Treat all orderings of the *same* set of factors as one way.

(A) 139 (B) 196 (C) 219 (D) 784 (E) None of the above

Problem 21. The harmonic triangle (a variation of Pascal's triangle) has the following properties:

(i) The n^{th} row has n entries, the first and last of which are both $1/n$.

(ii) Each entry is the sum of the two entries immediately *below* it.

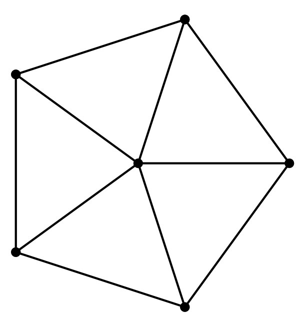
The first few rows of the harmonic triangle look like this:

$$
\begin{array}{c}
1\\
1/2 \quad 1/2\\
1/3 \quad 1/6 \quad 1/3\\
1/4 \quad 1/12 \quad 1/12 \quad 1/4\\
1/5 \quad 1/20 \quad 1/30 \quad 1/20 \quad 1/5\\
\vdots
\end{array}
$$

The outermost entries are the terms of the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, which is known to diverge. What is the sum of all the terms in the shaded region?

(A) 1/2 (B) 1 (C) 7/6 (D) 3/2 (E) The sum is divergent.

Problem 22. A rabbit jumps randomly from vertex to vertex in the following graph, every time choosing the direction randomly.



After a million jumps, what is the probability that the rabbit will be at the center of this graph?

(A) 1/6 (B) 1/5 (C) 1/4 (D) 1/3 (E) 2/5

Problem 23. A student is treated to a four-course meal, presented in random order. If the student eats a course and it is the best course he has eaten at that meal, he exclaims "Wow!" (For example, he will necessarily express delight after the first course.) What is the expected number of times that the student exclaims "wow"? Assume that no two courses are equally good.

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) None of the above

Problem 24. What is

$$\tan 20^\circ \tan 40^\circ \tan 80^\circ?$$

(A) $1/3$ (B) 1 (C) $\sqrt{2}$ (D) $\sqrt{3}$ (E) None of the above

Problem 25. Some 10 points are chosen in the plane. Mark the midpoints of all the intervals with endpoints at these 10 points. What is the smallest number of *distinct* points that can be marked?

(A) 9 (B) 17 (C) 27 (D) 45 (E) None of the above

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