Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 2, 2010

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

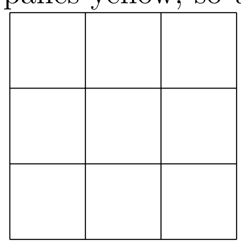
No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. What is the closest integer to

$$\frac{248163264}{123456}?$$

(A) 2 (B) 20 (C) 201 (D) 2010 (E) 201012

Problem 2. A window has 9 panes in the form of a 3×3 grid, as in the picture. In how many ways can one color 6 of these panes yellow, so that the window looks the same from inside and outside the house?



(A) 6 (B) 9 (C) 10 (D) 36 (E) None of the above

Problem 3. How many different ways are there to place seven rooks on a chessboard so that no two attack each other or occupy the same square?

Recall that a chessboard is an 8 by 8 grid. A rook attacks all the squares in the row *and* column that it occupies, a total of fifteen squares.

(A) 5040 (B) 40320 (C) 322560 (D) 362880 (E) None of the above

Problem 4. Recall that $\lceil x \rceil$ denotes the least integer greater than or equal to x. For what integer n do we have

$$\left\lceil \sqrt{1! + 2! + 3! + \cdots + n!} \right\rceil = 2010\,?$$

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Problem 5. Let

$$x = 1 + \frac{1}{2 + \frac{1}{3}}, \qquad y = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}, \qquad \text{and} \qquad z = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}}.$$

Which of the following is correct?

(A) $x < y < z$ (B) $x < z < y$ (C) $y < x < z$ (D) $y < z < x$ (E) $z < y < x$

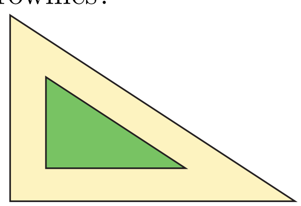
Problem 6. We all know that $\sin(0°) = \sin(0)$ (where the latter input is in radians). What is the smallest *positive* number x so that $\sin(x°) = \sin(x)$?

(A) π (B) 180 (C) $\dfrac{360\pi}{180-\pi}$ (D) $\dfrac{180\pi}{180+\pi}$ (E) $\dfrac{360\pi}{180+\pi}$

Problem 7. A phone number is *cool* if it is either of the form *abc-abcd* or of the form *abc-dabc* (or both) for some digits $a \neq 0$, b, c, and d. If numbers are assigned randomly, what is the chance that you will get a cool phone number? (Note: For the purposes of this problem, the first digit of any phone number cannot be 0, but there is no such restriction on the remaining digits.)

(A) 1/2 (B) $2010/10^6$ (C) $1999/10^6$ (D) 1/500 (E) None of the above

Problem 8. Derek prefers his brownies from the center of the pan, and Jacob prefers them from around the edge. Their friend Ellie gives them a pan of brownies in the form of a 3-4-5 right triangle. How far from the edges should Derek and Jacob cut it so that they each get equal areas of brownies?

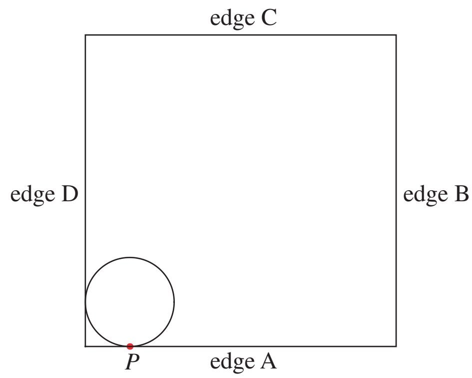


(A) $1 - 1/\sqrt{2}$ (B) 1/3 (C) $1/\sqrt{2}$ (D) 2/3 (E) 1

Problem 9. If q, r, and s are the solutions of $x^3 - 5x^2 + 7x + 4 = 0$, then what is $q(r^2+s^2) + r(s^2+q^2) + s(q^2+r^2)$?

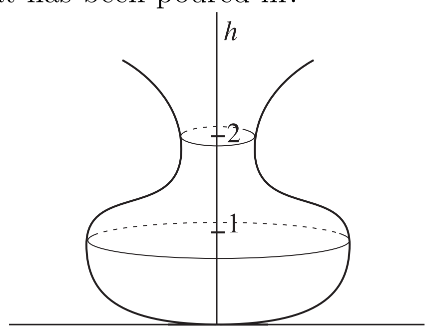
(A) -47 (B) -23 (C) 23 (D) 47 (E) not enough information

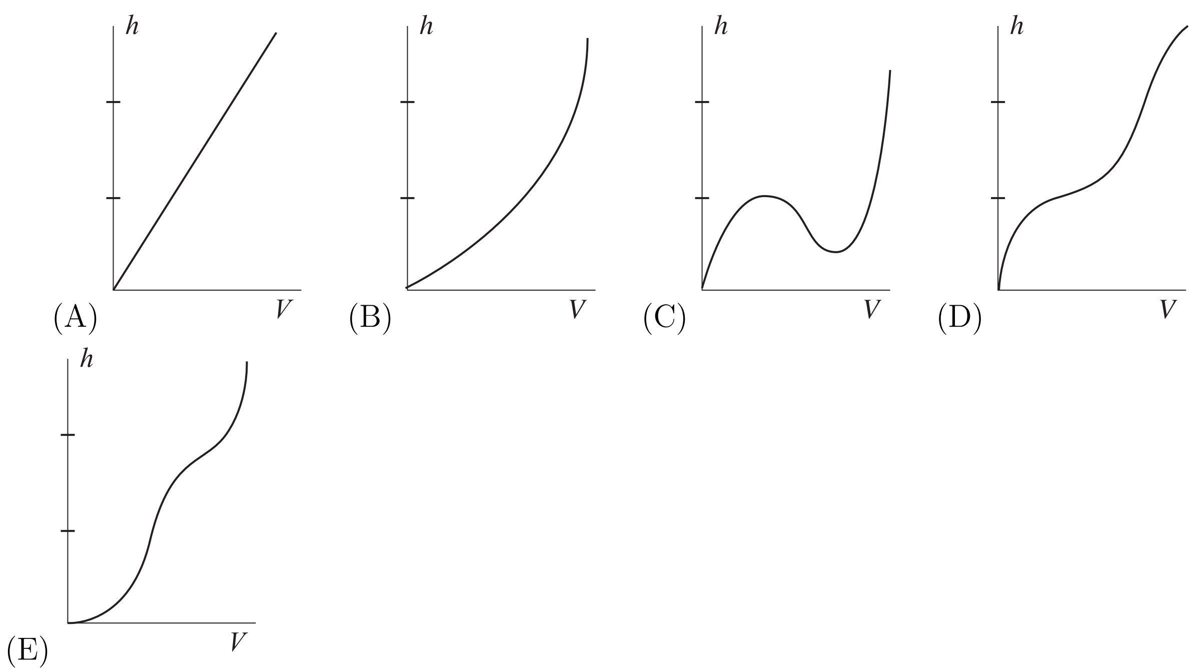
Problem 10. A circle of radius 1 is sitting inside a 7×7 square, tangent to the left and bottom edges, as pictured, with point P at the point of contact with edge A. If the circle rolls around the inside of the square without slipping, then which edge does the point P next touch?



(A) A (B) B (C) C (D) D (E) It never touches an edge again.

Problem 11. A vase in the shape shown below is slowly filled with water. Which of the graphs below most closely represents the height h of the water as a function of the volume V of water that has been poured in?





(A) (B) (C) (D) (E)

Problem 12. Let a_n be the n-th smallest positive integer the sum of whose decimal digits is 3. For example, $a_{18} = 2010$. How many digits does a_{1000} have?

(A) 15 (B) 16 (C) 17 (D) 18 (E) 19

Problem 13. Let $\sigma(n)$ be the sum of the positive divisors of n. For example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. How many integers n are there so that $\sigma(n) = 72$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 14. How many different Hamiltonian cycles are there on the vertices of the cube? That is, how many different ways are there to order the vertices of the cube as a cycle $(v_1, \ldots, v_8)$ so that consecutive vertices are adjacent (including v_8 and v_1)? Note that any cycles having the same adjacent vertices are considered the same: e.g., $(v_1, v_2, \ldots, v_8)$, $(v_2, v_3, \ldots, v_8, v_1)$, $(v_3, \ldots, v_8, v_1, v_2)$, etc., as well as $(v_8, \ldots, v_1)$, $(v_7, \ldots, v_1, v_8)$, etc.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 6

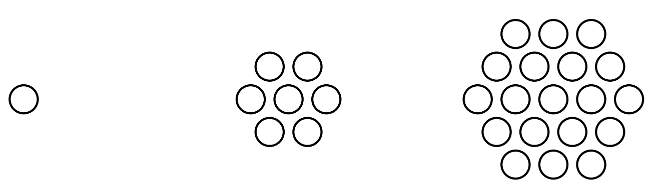
Problem 15. Let S be the sum of all seven-digit numbers whose digits are some permutation of 1, 2, 3, 4, 5, 6, and 7. Find the next-to-last ("tens") digit of S.

(A) 0 or 1 (B) 2 or 3 (C) 4 or 5 (D) 6 or 7 (E) 8 or 9

Problem 16. Suppose there are 10 points on the circumference of a circle. Draw all $\binom{10}{2} = 45$ chords connecting these points. What is the largest number of regions into which these chords can divide the circle?

(A) 128 (B) 256 (C) 512 (D) 1024 (E) None of the above

Problem 17. The first three centered hexagonal numbers ($h_1 = 1$, $h_2 = 7$, and $h_3 = 19$) are illustrated below:



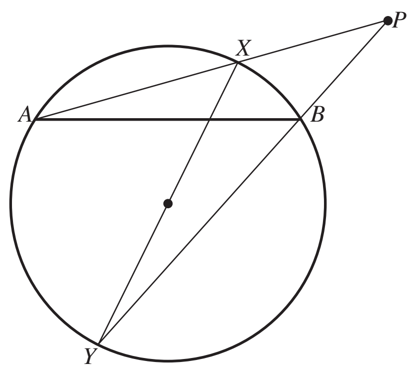
That is, the n-th centered hexagonal number h_n is the number of circles in a diagram that has one circle surrounded by $(n-1)$ layers of circles in a hexagonal lattice. What is the 10-th centered hexagonal number h_{10}?

(A) 217 (B) 231 (C) 276 (D) 331 (E) None of the above

Problem 18. What is the smallest positive integer that can **not** be written as the sum of 10 or fewer factorials?

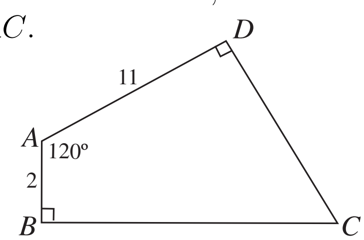
(A) 119 (B) 163 (C) 239 (D) 241 (E) $10! - 1$

Problem 19. Points A and B are fixed on a circle, and $\overline{AB}$ is not a diameter. Consider a diameter $\overline{XY}$ and the point P given by the intersection of $\overleftrightarrow{AX}$ and $\overleftrightarrow{BY}$, as pictured. What is the locus of all such points P as X moves all the way around the circle? (Note: When $A = X$, we have $P = A$, and when $B = Y$, we have $P = B$.)



(A) an arc of a circle (B) two arcs of a circle, not forming a complete circle (C) an ellipse that is not a circle (D) a circle (E) None of the above

Problem 20. Given quadrilateral $ABCD$, as pictured, with $\angle A = 120°$, and both $\angle B$ and $\angle D$ right angles. If $AB = 2$ and $AD = 11$, find AC.



(A) $5\sqrt{3}$ (B) $8\sqrt{3}$ (C) 14 (D) $10\sqrt{2}$ (E) None of the above

Problem 21. One marble is placed in each of three bowls. Five times in succession, a marble is moved from one bowl (chosen at random) to a different bowl (chosen at random). What is the probability that we again have one marble in each of the three bowls?

(A) 5/108 (B) 1/18 (C) 1/6 (D) 5/32 (E) 1/9

Problem 22. Now, *three* marbles are placed in each of three bowls. Five times in succession, a marble is moved from one bowl (chosen at random) to a different bowl (chosen at random). What is the probability that we again have three marbles in each of the three bowls?

(A) 5/108 (B) 1/18 (C) 1/6 (D) 5/32 (E) 1/9

Problem 23. Suppose triangle ABC has side lengths $BC = 13$, $AC = 14$, and $AB = 15$. Extend the two sides meeting at vertex A by BC, the two sides meeting at vertex B by AC, and the two sides meeting at vertex C by AB. The endpoints of these six new line segments all lie on a circle. The radius of this circle is $\sqrt{n}$ for some integer n. What is the sum of the digits of n?

(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Problem 24. In how many ways can 2010 be written as a sum of 2 or more *consecutive* positive integers? (For example, $9 = 4 + 5 = 2 + 3 + 4$ can be so written in 2 ways.)

(A) 1 (B) 3 (C) 5 (D) 7 (E) 8

Problem 25. Let $V_n(r)$ be the (n-dimensional) volume of the n-dimensional ball of radius r. So, $V_2(r) = \pi r^2$, $V_3(r) = \frac{4}{3}\pi r^3$, $V_4(r) = \frac{\pi^2}{2} r^4$, etc. We also set $V_0(r) = 1$. Find the sum

$$S = V_0(1) + V_2(1) + V_4(1) + \dots$$

(A) e^π (B) π^e (C) π^2 (D) infinite (the sum diverges) (E) None of the above

Authors. Jacob Rooney contributed 4, Mo Hendon contributed 5, 6, 8, 18, 24, Jacob Rooney and Derek Ponticelli contributed 22.

Solutions for 4, 5, 6, 8, 18, 19, 20, 24 were written by Ted Shifrin.

Problems and solutions for 1, 3, 12, 13, 14, 15, 16, 17, 21, 23, 25 are by Boris Alexeev, with contributions by Valery Alexeev.