1785

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 15, 2011

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

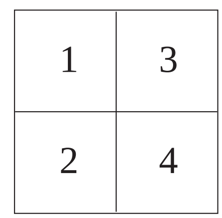
7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. Among families with exactly two children, at least one of which is a boy, what proportion have two boys? (Assume that genders of children are independent and equally likely to be male and female.)

(A) 1/4 (B) 1/3 (C) 1/2 (D) 2/3 (E) 3/4

Problem 2. A square piece of paper is subdivided into four congruent squares by a horizontal line and a vertical line, as pictured. The smaller squares are labeled 1, 2, 3, and 4, as shown. You are allowed to fold the paper once over the vertical line—left to right or right to left—and once over the horizontal line—top to bottom or bottom to top—in either order. After two folds, the smaller squares will be stacked up, and, if we read top to bottom, we get the numbers in some order. How many different permutations of 1, 2, 3, and 4 can we get?



(A) 2 (B) 4 (C) 8 (D) 12 (E) 24

Problem 3. Suppose $f(x)$ and $g(x)$ are quadratic polynomials, *all* of whose coefficients are *nonzero*. What is the minimum possible number of *nonzero* coefficients that $f(x)g(x)$ can have?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 4. A long row of doors, numbered 1, 2, 3, . . . , 100, are all open. Eddie comes along and closes every other door, beginning with the first. Eleanor closes every third door, beginning with the first. Note: Any door that is already closed remains so. This continues until the twenty-fourth person, Eli, closes every twenty-fifth door, beginning with the first. Clearly, the second door is still open. What is the next numbered door that is open?

(A) 27 (B) 28 (C) 29 (D) 30 (E) 31

Problem 5. Four distinct points A, B, C, and D are chosen at random from 2011 points evenly spaced on a circle. What is the probability that $\overline{AB}$ and $\overline{CD}$ intersect?

(A) 1/4 (B) 1/3 (C) 680/2011 (D) 1/2 (E) 3/4

Problem 6. What is the slope of the line that bisects the angle in the first quadrant formed by the x-axis and the line through the origin with slope 2?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{\sqrt{5}-1}{2}$ (D) 4/3 (E) $\frac{\sqrt{5}+1}{2}$

Problem 7. Let $O(n)$ denote the sum of the *odd* digits of n (by this we mean the digits of the numeral that are odd numbers, not the ones in odd decimal places). What is the sum

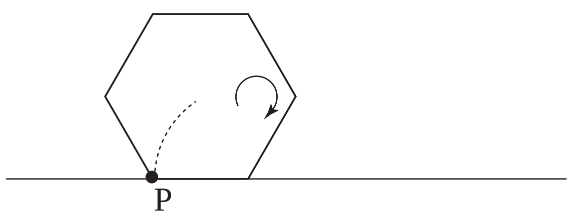
$$O(1) + O(2) + \cdots + O(100)\,?$$

(A) 251 (B) 276 (C) 500 (D) 501 (E) None of the above

Problem 8. How many monomials $x^a y^b z^c w^d$ are there with the requirement that a, b, c, d are nonnegative integers that sum to 10?

(A) 42 (B) 264 (C) 286 (D) 1001 (E) None of the above

Problem 9. A regular hexagon, as shown, with sidelength 1 "rolls" along a line. What is the length of the path that the vertex P travels as the hexagon rolls through one full cycle?



(A) π (B) 6 (C) $\frac{\pi}{3}(4+\sqrt{3})$ (D) $\frac{2\pi}{3}(2+\sqrt{3})$ (E) $2\pi\left(1+\frac{1}{\sqrt{3}}\right)$

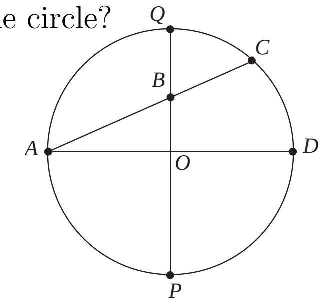
Problem 10. Suppose $3n$ has 56 positive divisors, $6n$ has 70 positive divisors, and $9n$ has M positive divisors. How many different possible values of M are there (that can be achieved by some positive integer n)?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 11. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a real polynomial *all* of whose coefficients are nonnegative integers. Suppose $f(1) = 17$ and $f(20) = 496{,}145$. What is a_3?

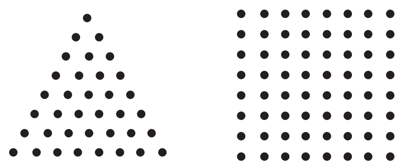
(A) 0 (B) 1 (C) 2 (D) 3 (E) 8

Problem 12. In the figure pictured, $\overline{AD}$ and $\overline{PQ}$ are diameters that are perpendicular to one another. If $AB = 8$ and $BC = 5$, what is the area of the circle?



(A) 30π (B) 36π (C) 45π (D) 52π (E) 56π

Problem 13. The number 36 has the unusual property that 36 dots can be arranged to form either an equilateral triangle or a square:



If a and b are the next larger integers with this property, what is $\sqrt{ab}$?

(A) 1225 (B) 2011 (C) 7140 (D) 44100 (E) There are no more numbers with this property.

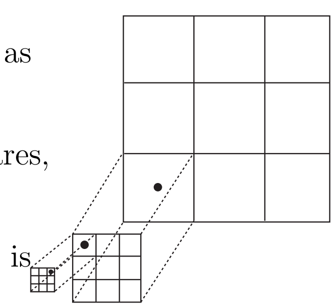
Problem 14. Among families with exactly two children, at least one of which is a boy born on a Tuesday, what proportion have two boys? (Assume as before that genders of children are independent and equally likely to be male and female, and similarly for day of the week of birth.)

(A) 1/3 (B) 4/7 (C) 13/27 (D) 1/2 (E) 147/196

Problem 15. Suppose $x^2 + y^2 + 6x - 4y - 12 = 0$. What is largest value that $2x + y$ can have?

(A) $4\sqrt{5} - 4$ (B) 5 (C) 7 (D) $5\sqrt{5} - 4$ (E) $4\sqrt{5}$

Problem 16. Consider the point $P = (a, b)$ with $0 \leq a \leq 1$, $0 \leq b \leq 1$. We picture it in the unit square. It has the following properties:



(i) If the square is subdivided into 9 smaller squares, as pictured, then P is in the lower left square.

(ii) If the lower left square is subdivided into 9 smaller squares, P is in the upper left square.

(iii) If that square is subdivided into 9 smaller squares, P is in the upper right square.

(iv) If that square is subdivided into 9 smaller squares, P is in the lower right square.

The subdivision process continues like this, and P continues to be in the lower left, upper left, upper right, lower right squares, forever. What is $a + b$? (The square is oriented so that a increases from left to right and b increases from bottom to top, as usual.)

(A) 1012/9999 (B) 1/3 (C) 10/27 (D) 2/5 (E) 1/2

Problem 17. Notice that the graphs $y = e^x$ and $y = x$ do not intersect in the first quadrant. If a is the smallest positive number so that $y = e^x$ and $y = x^a$ *do* intersect in the first quadrant, which of the following is true?

(A) There is no smallest a because $y = e^x$ intersects $y = x^a$ for all $a > 1$. (B) $a = 2$ (C) $a = e$ (D) $a = \pi$ (E) There is no smallest a because $y = e^x$ fails to intersect $y = x^a$ for all $a > 1$.

Problem 18. Five circles of radius 1 are packed tightly so that their centers form the vertices of a regular pentagon. Which of the following is closest to the radius of the smallest circle that will contain them all?

(A) 2.5 (B) 2.6 (C) 2.7 (D) 2.8 (E) 3

Problem 19. Five circles of the same radius are packed in a square of unit side length. (That is, the interiors of the circles lie strictly within the square and do not intersect each other. Note that they are not necessarily packed exactly as in the previous problem.) What is the largest possible radius of the circles?

(A) $\frac{2}{4+3\sqrt{2}+\sqrt{6}}$ (B) $\frac{1}{5}$ (C) $\frac{(\sqrt{2}-1)}{2}$ (D) $\frac{1}{4}$ (E) None of the above

Problem 20. Consider a Pascal-like triangle with the numbers 0, 1, 2, 3, ..., going down the edges, as shown:

$$\begin{array}{ccccccccccc} &&&&& 0 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 2 && 2 && 2 &&& \\ && 3 && 4 && 4 && 3 && \\ & 4 && 7 && 8 && 7 && 4 & \\ 5 && 11 && 15 && 15 && 11 && 5 \end{array}$$

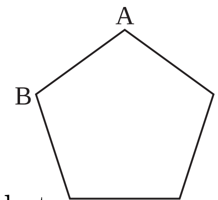
Let $f(n)$ denote the sum of the entries in the row that begins with n. When we divide

$$\frac{f(100)}{f(50)}$$

by 25, what is the remainder?

(A) 0 (B) 2 (C) 3 (D) 24 (E) None of the above

Problem 21. You start at vertex A of a pentagon. It is equally likely that you move along either of the adjacent edges. You continue, with either of the adjacent edges always having equal probability. You stop as soon as you've visited each vertex. What is the probability that you stop at the adjacent vertex, B? (Remark: You can assume the tour stops after a finite number of steps.)



(A) 1/6 (B) 1/5 (C) 1/4 (D) 5/18 (E) 1/3

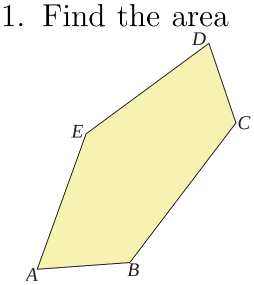
Problem 22. There exist two *distinct* positive integers x and y so that

$$\left(\frac{x}{82}\right)^2 + \left(\frac{y}{82}\right)^2 = 2.$$

What is $|x - y|$?

(A) 18 (B) 24 (C) 36 (D) 62 (E) None of the above

Problem 23. A pentagon $ABCDE$, as pictured, has the property that each of the triangles $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEA$, and $\triangle EAB$ has area 1. Find the area of the entire pentagon.



(A) $\dfrac{3+\sqrt{5}}{2}$ (B) 3 (C) $2+\dfrac{\sqrt{5}}{2}$ (D) $\dfrac{5+\sqrt{5}}{2}$ (E) 4

Problem 24. Andy and Harrison are betting on a seven-game series between the Astronauts and the Hedgehogs. Before each game, they agree to an even-odds bet for some specified amount (possibly zero, possibly much more) on that single game. (The amount of the bet will depend on the score.)

They have structured their bet amounts in advance so that although they are betting on individual games, the sequence of bets is equivalent to a single \$1 bet on the entire series. That is, no matter with what score the Astronauts win (4-0, 4-1, 4-2, or 4-3), Andy will win exactly \$1, and similarly if the Hedgehogs win, Harrison will win exactly \$1.

How much are they betting on the first game? (Assume, if necessary, that money is infinitely divisible.)

(A) $\$\frac{5}{16}$ (B) $\$\frac{1}{7}$ (C) $\$\frac{1}{4}$ (D) $\$\frac{1}{14}$ (E) \$0

Problem 25. The Battle of Hastings (October 14, 1066): "The men of Harold stood well together, as their wont was, and formed nineteen squares, with a like [positive] number of men in every square thereof, and woe to the hardy Norman who ventured to enter their redoubts; for a single blow of a Saxon [warrior] would break his lance and cut through his coat of mail.... When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle-cries, 'Ut!' 'Olicross!', 'Godemite!'."

What is the smallest number of Saxons (counting Harold) that could have been at the battle?

(A) 15,876 (B) 28,900 (C) 31,940 (D) 33,856 (E) None of the above

Authors. Written by Ted Shifrin, with significant help from Mo Hendon. Alex Rice contributed problems 1 and 14, having learned of them from Colm Mulcahy. Kate Thompson, Derek Ponticelli, and Tyler Kelly contributed problems 25, 11, and 21 respectively. Boris Alexeev contributed problems 19, 22, 24.