Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 1 or 2 minutes per problem

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for the correct answer obtained in the first minute, and 5 points for the correct answer obtained in the second minute.

Problem 1. There are three hats with numbers 1, 2 and 3 written on them, and three balls also numbered 1, 2 and 3. In how many ways can one put the balls in the hats so that no ball gets into a hat with the same number? (One can put more than one ball into hats).

Answer.

$$2^3 = 8$$

Solution. Each of the 3 balls can be put in 2 possible hats.

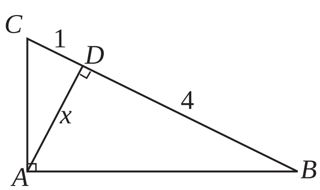
Problem 2. When the volume of a spherical balloon is multiplied by 3, by what factor does its surface area increase?

Answer.

$$(\sqrt[3]{3})^2 = \sqrt[3]{9} = 3^{2/3}$$

Solution. When the *radius* is increased by a factor x, the volume is increased by x^3 and surface area by x^2. So, $x^3 = 3$, $x = \sqrt[3]{3}$ and $x^2 = (\sqrt[3]{3})^2$.

Problem 3. Given right triangle $\triangle ABC$ with $\overline{AD} \perp \overline{BC}$, $BD = 4$ and $CD = 1$. Find AD.



Answer.

$$2$$

Solution. $\triangle ABD \sim \triangle CAD$, so

$$\frac{x}{1} = \frac{4}{x},$$

from which we get $x^2 = 4$, so $x = 2$.

Problem 4. What is the area of a regular 12-sided polygon (dodecagon) inscribed in a circle with radius 1?

Answer.

$$3$$

Solution. It splits into 12 triangles of area

$$\frac{1}{2}1 \cdot 1 \cdot \sin 30^\circ = 1/4$$

so the total area is $12/4 = 3$.

Problem 5. Find the sum of the prime factors of 1591.

Answer.

$$80$$

Solution. Indeed,

$$1591 = 1600 - 9 = 40^2 - 3^2 = 43 \cdot 37$$

and the sum of prime factors is $43 + 37 = 80$.

Problem 6.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{24 \cdot 25} =$$

Answer.

$$\frac{24}{25}$$

Solution. We have
$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1},$$
so
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{24\cdot 25} =$$
$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{23}-\frac{1}{24}\right)+\left(\frac{1}{24}-\frac{1}{25}\right) = 1-\frac{1}{25} = \frac{24}{25}.$$

Problem 7. What is the radius of the smallest sphere containing the spheres with equations
$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad (x-1)^2 + (y-2)^2 + (z-2)^2 = 4?$$

Answer.
$$3$$

Solution. The distance between the centers of the spheres is $\sqrt{1^2 + 2^2 + 2^2} = 3$, and radii are 1 and 2. So the radius of the bigger sphere is $(3+1+2)/2 = 3$.

Problem 8. How many digits are in the base-ten numeral $4^{22} \cdot 5^{40}$?

Answer.
$$42$$

Solution. We have
$$4^{22} \cdot 5^{40} = 2^{44} \cdot 5^{40} = 2^4 \cdot 10^{40} = 16 \cdot 10^{40},$$

so there are $2 + 40 = 42$ digits.

Problem 9. Determine the missing digit $\diamondsuit$ in the following mystery multiplication problem:

$$\begin{array}{ccccccccc} & & & 4 & 3 & ? & ? & 8 & ? \\ & & & & \times & & 7 & 5 & 6 \\ \hline 3 & 3 & 1 & \diamondsuit & 6 & 8 & 6 & 1 & 6 \end{array}$$

Answer.

$$2$$

Solution. The product must be divisible by 9 because 756 is. Remember that a number is divisible by 9 precisely when the sum of its digits is divisible by 9 ("casting out nines"). The digits without $\diamondsuit$ add up to $7 \pmod 9$, so $\diamondsuit$ must be a 2.

Problem 10. Alice, Bob and Caroline compete who can solve more ciphering problems. In how many different orders can they finish if ties can happen? (For example, for two people there are 3 possible outcomes)

Answer.

$$13$$

Solution. There is 1 outcome where all tied. There are 6 outcomes with no ties. In addition, there are 3 outcomes where the top 2 people tie, and 3 outcomes where the bottom 2 people tie. Altogether,

$$1 + 6 + 3 + 3 = 13$$