University of Georgia High School Math Tournament

1785

2005

Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem

WITH SOLUTIONS

No calculators are allowed on this test. 2 minutes per problem, 10 points for each correct answer.

Problem 1. In a barn with chickens and dogs there are 5 heads and 14 legs. How many chickens are there? (A chicken has 2 legs and a dog has 4.)

Answer. 3

Solution. If all dogs stand on 2 legs, there will be 10 legs on the floor. Once the dogs stand on all 4 legs, there will be 14 legs, 4 more. So, there are 2 dogs and hence 3 chickens.

Problem 2. Several logs are cut into 16 pieces by making a total of 10 cuts (every time only one log is cut). How many logs were there?

Answer. 6

Solution. Each cut adds one piece, so there were 6 logs.

Problem 3. Ted drives to Atlanta at 60 mph and returns at 30 mph. What was his average speed for the round trip, in mph?

Answer. 40

Solution. Denote the distance between Athens and Atlanta by d (miles). Then it will take Ted $d/60 + d/30$ hours to make the roundtrip, and the average speed will be

$$\frac{2d}{\frac{d}{60} + \frac{d}{30}} = \frac{2}{\frac{3}{60}} = \frac{120}{3} = 40 \text{ mph}$$

Problem 4. Express $\sqrt{3-4i}$ in the form $a+bi$ with $a > 0$. (Here, $i = \sqrt{-1}$.)

Answer.

$$2 - i$$

Solution.

$$(a+bi)^2 = (a^2 - b^2) + 2abi$$

So, $a^2 - b^2 = 3$ and $2ab = 4$, $ab = -2$. Obviously, $a = 2$, $b = -1$ works.

Problem 5. In the alphabet of the Mumbo-Jumbo tribe there are 3 letters. A word is any sequence of these letters which is 4 letters or shorter. How many words are there in the language of Mumbo-Jumbo?

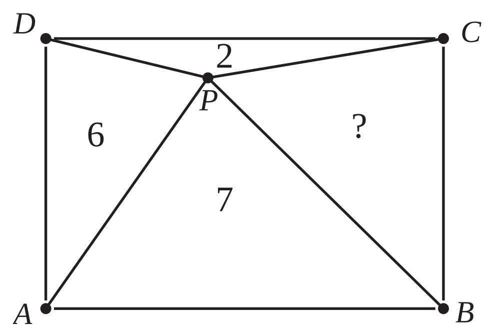
Answer.

120

Solution.

$$3 + 3^2 + 3^3 + 3^4 = 3(1 + 3 + 3^2 + 3^3) = 3(1 + 3 + 9 + 27) = 120$$

Problem 6. Point P is inside rectangle $ABCD$. In sq. units, the areas of $\triangle APB$, $\triangle APD$, and $\triangle CPD$ are 7, 6, and 2, respectively. Find the area of $\triangle BPC$.



Answer. 3

Solution. $9 = \text{area}\triangle APB + \text{area}\triangle CPD = \frac{1}{2}(AB)(BC)$. But this is also $\text{area}\triangle BPC + \text{area}\triangle APD$, so $\text{area}\triangle BPC = 9 - 6 = 3$.

Problem 7. How many 6-digit numbers are divisible by 5?

Answer.

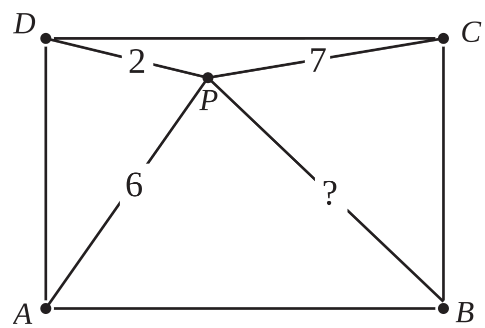
$$180,000$$

Solution. There are 9 possibilities for the first digit: 1–9, 10 possibilities for digits two through 5, and 2 possibilities for the last digit: 0 and 5. Therefore, there are

$$9 \cdot 10^4 \cdot 2 = 180,000$$

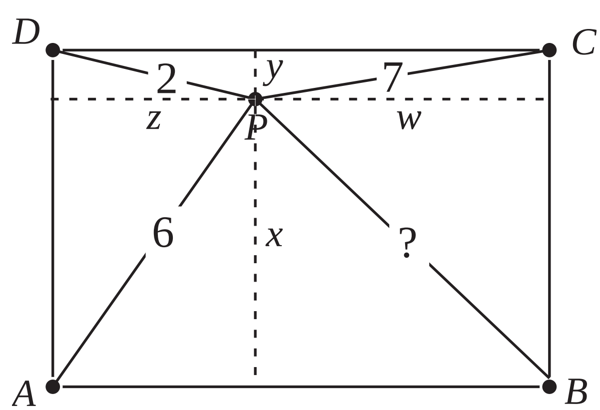
numbers in all.

Problem 8. Point P is inside rectangle $ABCD$. $AP = 6$, $DP = 2$, and $CP = 7$. Find BP.



Answer. 9

Solution.



We have

$$\begin{aligned} x^2 + z^2 &= 36 \\ y^2 + w^2 &= 49 \\ y^2 + z^2 &= 4 \\ x^2 + w^2 &= ?^2 \end{aligned}$$

Therefore, $?^2+4 = (x^2+w^2)+(y^2+z^2) = (x^2+z^2)+(y^2+w^2) = 36+49 = 85$, so $? = 9$.

Problem 9. How many zeros are at the end of the base three decimal for 27! ?

Answer. 13

Solution. $27 = 3^3$, 9 and 18 are divisible by 3^2, 3,6,12,15,21,24 are divisible by 3^1. Together, this gives

$$3 + 2 \cdot 2 + 6 = 13$$

Problem 10. What is the smallest integer $n > 2$ for which the fraction

$$\frac{n-2}{n^2+13}$$

is **not** in lowest terms?

Answer. 19

Solution. The fraction fails to be in lowest terms if and only if there is some prime p that divides both numerator and denominator. This occurs if and only if $n - 2 \equiv 0 \pmod{p}$ and $n^2 + 13 \equiv 0 \pmod{p}$, so $n \equiv 2$ and therefore $17 \equiv 0 \pmod{p}$. This means that $p = 17$ and so the smallest n is 19.

 Some problems were taken from N.B. Alfutova, A.B. Ustinov "Algebra and number theory for mathematical schools" published by Moscow Center for Continuing Mathematical Education, 2002.