Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 15, 2011

WITH SOLUTIONS

No calculators are allowed on this test. 2 minutes per problem, 10 points for each correct answer.

Problem 1. The length of a rectangle increases by 20% and its width decreases by 10%. By what percentage does the area of the rectangle increase?

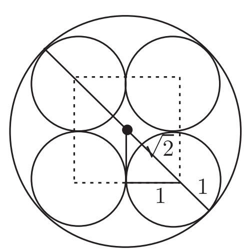
Answer. 8

Solution. Let x denote the original length, y the original width. Then the new rectangle has area $(1.2x)(0.9y) = 1.08xy$. This means the area increases by 8%.

Problem 2. Four circles with radius 1 are packed tightly together. What is the radius of the smallest circle that will contain them all?

Answer. $1 + \sqrt{2}$

Solution. See the figure.



Problem 3. Find *all* real solutions of the equation $x + 4\sqrt{x} - 21 = 0$.

Answer. 9

Solution. Let $\sqrt{x} = u$. Then we have $u^2 + 4u - 21 = (u+7)(u-3) = 0$, so $u = -7$ or $u = 3$. Since $\sqrt{x} = u$ must be nonnegative, we must have $u = 3$ and $x = 9$.

Problem 4. Four identical tennis balls are packed, one on top of the other, tightly (but without changing their spherical shape) in a cylindrical can. What fraction of the volume of the can is outside the balls?

Answer. 1/3

Solution. Interestingly, the answer is independent of the number of balls packed into the cylindrical can of the appropriate size. If a tennis ball has radius r, then we need a can of radius r and height $8r$, which has volume $8\pi r^3$. The total volume of the balls is $4 \cdot \frac{4}{3}\pi r^3 = \frac{16}{3}\pi r^3$. Since $\frac{16/3}{8} = \frac{2}{3}$, one-third the can is outside the balls.

Problem 5. Derek has in his pocket assorted coins (some combination of pennies, nickels, dimes, quarters, and fifty-cent pieces). What is the largest possible amount of money he can have without being able to make change for a nickel, a dime, a quarter, a fifty cent piece or a dollar?

Answer. \$1.19

Solution. Derek can have 4 pennies, 4 dimes, 1 quarter, and 1 fifty-cent piece, totalling \$1.19. (It is easy to make the mistake of allowing him only 2 dimes.)

Problem 6. Justin, Miley, and Charice are among eight singers who will divided at random into two groups of four. What is the probability that all three end up in the same group?

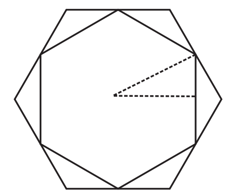
Answer. 1/7

Solution. (Corrected Version) There are 5 possible groups of four containing all three stars. There are $\binom{8}{4} = 70$ different groups of four, but only $70/2 = 35$ complementary *pairs* of groups. The likelihood that the three stars end up together in one of those pairs is $5/35 = 1/7$.

Note: an incorrect answer 1/14 was initially provided. We've given credit for both 1/7 and 1/14.

Problem 7. The midpoints of a regular hexagon are joined to form another regular hexagon inside. What is the ratio of the area of the inner hexagon to the area of the outer hexagon?

Answer. 3/4



Solution. See the figure. The inner hexagon is similar to the outer, and the ratio of their altitudes is $\cos(\pi/6) = \sqrt{3}/2$. Therefore, the ratio of their areas is $(\sqrt{3}/2)^2 = 3/4$.

Problem 8. What is the minimum value of

$$x^2 + y^2 + x - 4y + 5$$

as x and y range over all real numbers?

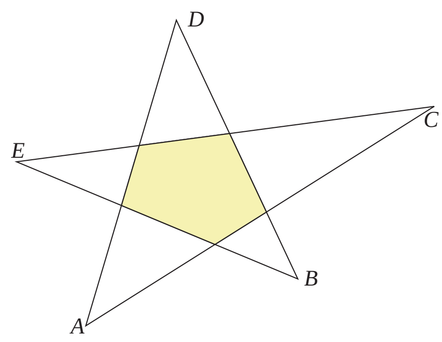
Answer. 3/4

Solution. Completing the square, we have

$$\begin{aligned} x^2 + y^2 + x - 4y + 5 &= \left((x + \tfrac{1}{2})^2 - \tfrac{1}{4}\right) + \left((y-2)^2 - 4\right) + 5 \\ &= (x + \tfrac{1}{2})^2 + (y-2)^2 + \tfrac{3}{4}\,. \end{aligned}$$

Since the square of any real number is always nonnegative, we see that 3/4 is the smallest possible value.

Problem 9. In radians, what is the sum $\angle A + \angle B + \angle C + \angle D + \angle E$ in the figure?



Answer. π

Solution. We know that the exterior angles of any polygon (in this case, the shaded pentagon) sum to 2π. That sum appears twice when we consider the five "star" triangles. So the sum of the angles at the marked vertices is $5 \cdot \pi - 2 \cdot 2\pi = \pi$.

Problem 10. Given a set A of real numbers, define $A + A = \{a + b : a, b \in A\}$. For example, if $A = \{1, 4\}$, then $A + A = \{2, 5, 8\}$. If A consists of precisely *four* (different) numbers, what is the smallest number of elements in $A + A$?

Answer. 7

Solution. In principle, there could be 10 possible sums (down from 16 because addition is commutative). Consider the set $A = \{0, 1, 2, 3\}$. This gives $A + A = \{0, 1, 2, 3, 4, 5, 6\}$, which has seven elements. Without loss of generality, assume the smallest element is 0 and the remaining elements are x, y, and z, with $x < y < z$. Since $\{0, x, y, z\} \subset A+A$, we do best if $y = 2x$ and $z = x+y$; this eliminates two more of the 10 possible sums. But then we eliminate one more for free, since $2y = x + z$. We can't do any better than this.

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