1785

Sponsored by: UGA Math Department and UGA Math Club

Team Round / 45 min / 210 points
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WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1. (Up or down?) What is the smallest n such that in any sequence of n distinct numbers

$$a_1, a_2, a_3, \dots, a_n$$

there is either an increasing subsequence of length 10 or a decreasing subsequence of length 10?

Example. In the sequence

$$1, 3, 2, 8, 4, 0, 5$$

a longest increasing sequence has length 4:

$$\boxed{1}, 3, \boxed{2}, 8, \boxed{4}, 0, \boxed{5}$$

and a longest decreasing sequence has length 3:

$$1, \boxed{3}, \boxed{2}, 8, 4, \boxed{0}, 5$$

Answer. 82

Solution. *Hint:* you can guess the answer by looking at the easier cases of sequences of lengths 2, 3, 4.... The answers are 2, 5, 10...

Here is a sequence of longest possible length 81 which *does not* have increasing or decreasing subsequences of length 10:

$$9, 8, \ldots, 1,\ 18, 17, \ldots, 10,\ 27, 26, \ldots, 19, \ldots, 81, 80, \ldots, 73.$$

But we claim that every sequence of length ≥ 82 will have a required subsequence. Here is why. To every element a_i of the sequence, assign a pair of numbers (b_i, c_i) of positive integers as follows:

1. b_i, the length of the longest increasing sequence ending with a_i, and

2. c_i, the length of the longest decreasing sequence ending with a_i.

We claim that for different a_i and a_j, the pairs are different.

Indeed, let us say $i < j$. If $a_i < a_j$ then the longest increasing sequence for a_j is guaranteed to be longer than the one for a_i, so $b_j > b_i$. Similarly, if $a_i > a_j$ then $c_j > c_i$.

If all increasing sequences have lengths ≤ 9 then for the pairs (b_i, c_i) there are only $9^2 = 81$ possibilities. So with 82 numbers, we are guaranteed to have a sequence of length 10.

Problem 2. (Doubling up) Find a positive integer which doubles when its last digit is moved in front. The number is to be written in standard decimal notation with no leading zeroes.

Example. The number 1234 becomes 4123 when its last digit is moved in front (so it doesn't work).

Answer.

$$105263157894736842$$

is the smallest. One can obtain every such number from this by rotating the digits (in such a way that the next to the last digit is even) or by repeating the entire number multiple times, *e.g.*

$$947368421052631578947368421052631578.$$

(So the number of digits is divisible by 18.)

Solution. *First solution.* Let x be such a number, with last digit d and n digits total. Then by construction, $\frac{x-d+10^n d}{10} = 2x$, or in other words

$$19x = (10^n - 1)d.$$

Thus we seek an n such that $10^n - 1$ is divisible by 19. By trying successive n or by Fermat's little theorem, $n = 18$ works. Setting $d = 1$ results in a leading 0, but $d = 2$ gives

$$\frac{10^{18} - 1}{19} \cdot 2 = 105263157894736842.$$

Second solution. One could also construct this number as follows.

You want the last digit as small as possible, so that the doubled number will start with a smal digit. But the last digit can't be 0 – moving that to the front makes the number smaller – and it can't be a 1 – a doubled number cannot begin with a 1 unless it has more digits than the number that was doubled. So begin with last digit 2. Moving this to the front is supposed to double the number V, so the last digit of the doubled number $2N$, and thus next to the last digit of N, is 4. Repeating this, we get

$$N = \ldots 6842$$

At this point we have to start paying attntion to carries: $2 \times 842 = 1684$, so the next digit must be 3 ($= 2 \times 6 + 1 \mod 10$). Continuing, we get

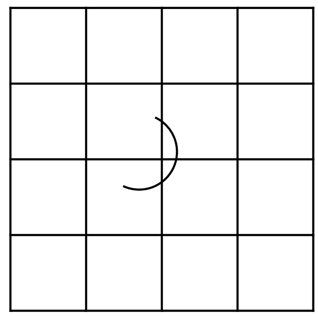
$$N = 105263157894736842$$

We could continue, but we would only repeat the same digits.

Beginning with any other digit $d > 2$ will generate the same set of digits as above, beginning with d which is preceded by an even digit.

Problem 3. (Lucky horseshoe) A horseshoe has the shape of a semicircle of diameter 1. We throw it randomly on a square grid of 1 by 1 squares and count how many times it intersects the lines. After a very large number N of throws, the number of intersections will be close to cN for some number c. What is c?

Example. The following throw counts for three intersections:



Answer. 2

Solution. If we throw a *circle* of diameter 1, then after N times, we expect approximately $4N$ intersections. Indeed, the circle will have exactly 4 intersections each time, except in the impossibly unlikely event that it hits one of the grid's intersection points. Two semicircles make a circle, so for each of them the number of intersections will be $2N$. So $c = 2$.

In general, for an arbitrary curve of length ℓ, one has $c = 4\ell/\pi$. To prove this, note that for the mathematical expectation (that is what we are computing here) one has

$$E(\xi_1 + \cdots + \xi_n) = E\xi_1 + \cdots + E\xi_n$$

Therefore, we can slice our curve into small pieces and make other curves from these pieces. If our curve has a length then it can be approximated by a sequence of straight segments, by which we can then approximate another curve. Then the above formula shows that the answer is proportional to the length of the curve. For a circe of diameter 1 and length π, the answer is 4. So the coefficient of proportionality is $4/\pi$.

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Sources. (1) is a special case of a theorem of Erdős-Szekeres, see `http://en.wikipedia.org/wiki/Erdos-Szekeres_theorem`. For (2), see this New York Times article about Freeman Dyson: `http://www.nytimes.com/2009/03/29/magazine/29Dyson-t.html`, last page. Gil Kalai recently blogged about problem (3) on his blog "Combinatorics and more": see the entry on `http://gilkalai.wordpress.com/2009/08/03/` about Buffon's needle and Buffon's "noodle".