Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 2, 2010

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

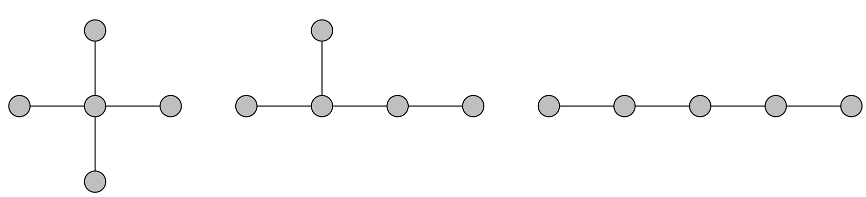
Problem 1 (Prüfer Airlines). In a certain small European country there are only 5 cities. An airline decides to connect them by 4 (two-way) connections, so that it would be possible to fly from any city to any other, possibly with stops.

(a) (35 points) In how many ways is it possible to do this?

(b) (35 points) Same question but for 6 cities and 5 connections.

Answer. (a) $125 = 5^3$, (b) $1296 = 6^4$

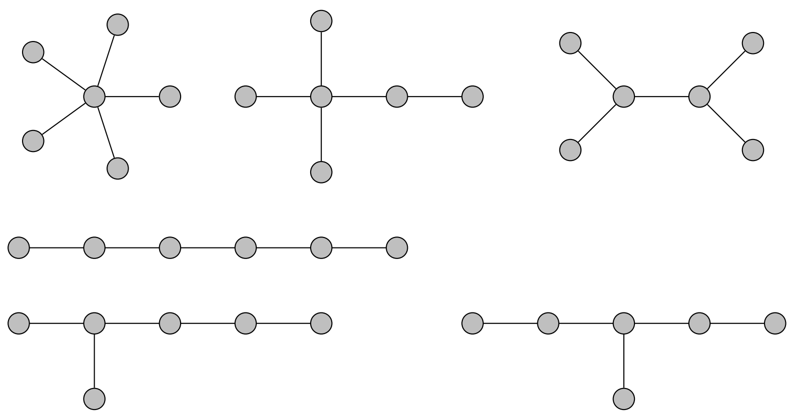
Solution. (a) Draw five vertices, label them by writing numbers 1 through 5, and then draw the 4 connections. What you get is called a *tree* in mathematics, a connected *graph* which has no cycles. For 5 vertices, the possibilities for the trees are given in the picture below.



For each of these pictures, we need to count the ways we can label the vertices with numbers 1 through 5.

For the first graph, there are 5 ways: 5 choices for the central vertex, after that we get the same result up to symmetries. For the second tree, there are $5 \cdot 4 \cdot 3 = 60$ ways. And for the third three, there are $\frac{5!}{2} = 60$ ways, for a total of 125.

(b) Similarly, for the case of 6 vertices, the possible pictures are given in the next picture.



Counting them, we get

$$6 + \frac{6!}{3!} + \frac{6!}{8} + \frac{6!}{2} + \frac{6!}{2} + \frac{6!}{2} =$$
$$6 + 120 + 90 + 360 + 360 + 360 = 1296$$

(For each tree, the number of ways is 6! divided by the number of symmetries of that tree.)

The general formula for n vertices is n^{n-2}. This is a famous theorem of Cayley from 1889. The easiest and most beautiful way to prove it is using Prüfer sequences (`http://en.wikipedia.org/wiki/Prufer_sequence`). One shows that every labeled tree can be uniquely encoded by a sequence of numbers 1 through n of length $n - 2$, with repetitions allowed. Of course, there are n^{n-2} such sequences.

For the details of the encoding and decoding procedures, see for example `http://www.proofwiki.org/wiki/Prufer_Sequence_from_Labeled_Tree`, `http://www.proofwiki.org/wiki/Labeled_Tree_from_Prufer_Sequence`.

Problem 2 (Let's be friends). One hundred (100) people go through the following procedure. One-by-one, they each randomly point at a person who is not yet pointed at. A person may point at himself, so for example, the first person points at himself with probability 1%.

What is the probability that after this procedure there exist 75 people $P_1, P_2, \ldots, P_{75}$ so that person P_1 points at P_2, person P_2 points at P_3, $\ldots$, person P_{74} points at P_{75}, and finally person P_{75} points at P_1?

Answer. $\frac{1}{75}$

Solution. The procedure given corresponds to choosing a random permutation of 100 elements, and the question is whether or not this permutation contains a 75-cycle.

Overall, there are 100! permutations. Let's count how many of them contain a 75-cycle. First of all, there is at most one 75-cycle. There are $\binom{100}{75}$ different ways of choosing which 75 people comprise the cycle, 74! different ways of choosing the order of the cycle, and 25! different ways of choosing what the other 25 people do. (It seems like there might be 75! different orders, but two orders are equivalent if you "rotate the cycle" in one of the 75 ways.) Thus the overall probability is

$$\binom{100}{75}\frac{74!25!}{100!} = \frac{1}{75}.$$

Generalization. The proof above applies for any cycle length greater than 50, but not for smaller cycles, because then you could have more than one cycle of the given length. The probability that there is a cycle of *any* length greater than 50 is thus

$$\frac{1}{51} + \cdots + \frac{1}{100},$$

again using that there is at most one long cycle (and subtly using the linearity of expectation without mentioning it). This is a Riemann sum for approximating the integral

$$\int_{\frac{1}{2}}^{1} \frac{dx}{x} = \log 2,$$

so the sum above is approximately $\log 2 \approx 0.693$. (The actual sum is about 0.688.)

Problem 3 (A very fair division). It is possible to divide the integers $1, 2, \ldots, 8$ into two sets A and B in a unique manner so that

- 1 is in A,
- A and B contain the same number of elements, and
- the sum of the elements in A equals the sum of the elements in B, and
- the sum of the squares of the elements in A equals the sum of the squares of the elements in B.

It is also possible to divide the integers $1, 2, \ldots, 16$ into two sets A and B in a unique manner so that all of the above hold, *as well as*

- the sum of the *cubes* of the elements in A equals the sum of the *cubes* of the elements in B.

This is a two-part problem:

(a) (35 pts) What is the set A in the case of 8 numbers?

(b) (35 pts) What is the set A in the case of 16 numbers?

Answer.

(a) $\{1, 4, 6, 7\}$

(b) $\{1, 4, 6, 7, 10, 11, 13, 16\}$

Solution. It is possible to build these sets up inductively. If you have a collection A and a collection B such that

$$\sum_{a \in A} a^i = \sum_{b \in B} b^i$$

for all i from 0 to $n-1$, then the two collections $A' = A \cup (B + x)$ and $B' = B \cup (A + x)$ satisfy

$$\sum_{a \in A'} a^i = \sum_{b \in B'} b^i$$

for all i from 0 to n, regardless of the choice of x.

Using this principle, you can build up the sets as follows:

- The sets $A_1 = \{1\}$ and $B_1 = \{2\}$ satisfy the above properties for $n = 1$. (That is, they have the same number of elements.)
- Thus the sets $A_2 = A_1 \cup (B_1 + 2) = \{1, 4\}$ and $B_2 = B_1 \cup (A_1 + 2) = \{2, 3\}$ satisfy the above properties for $n = 2$. (That is, they have the same number of elements, and the same sum.)
- Thus the sets $A_3 = A_2 \cup (B_2 + 4) = \{1, 4, 6, 7\}$ and $B_3 = B_2 \cup (A_2 + 4) = \{2, 3, 5, 8\}$ satisfy the above properties for $n = 3$. (That is, this is the first part of the problem.)
- Thus the sets $A_4 = A_3 \cup (B_3 + 8) = \{1, 4, 6, 7, 10, 11, 13, 16\}$ and $B_4 = B_3 \cup (A_3 + 8) = \{2, 3, 5, 8, 9, 12, 14, 15\}$ satisfy the above properties for $n = 4$. (That is, this is the second part of the problem.)

Let's prove the principle. We have

$$\begin{aligned}
\sum_{a' \in A'} (a')^i &= \sum_{a \in A} a^i + \sum_{b \in B} (b + x)^i \\
&= \sum_{a \in A} a^i + \sum_{j=0}^{i} \binom{i}{j} x^{i-j} \sum_{b \in B} b^j \\
&= \sum_{b \in B} b^i + \sum_{j=0}^{i} \binom{i}{j} x^{i-j} \sum_{a \in A} a^j \\
&= \sum_{b \in B} b^i + \sum_{a \in A} (a + x)^i = \sum_{b' \in B'} (b')^i,
\end{aligned}$$

where in the case $i < n$, we were able to use the inductive hypothesis to switch summations over A and B. In the case $i = n$, we do the same thing, except we are also able to switch the terms with exponent n because they are exactly the same. (This is not possible for the other terms, because one of them has coefficient 1 while the other has coefficient $\binom{i}{j} x^{i-j}$, which is not 1 if $i \neq j$.)

Connections. This pattern of As and Bs is connected to an infinite binary sequence called the Thue-Morse sequence:

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

(See `http://en.wikipedia.org/wiki/Thue-Morse_sequence`.)

In this sequence, a 0 in the nth position indicates that n is in A and a 1 indicates that n is in B. The sequence can be formed by taking the first 2^k elements, flipping them, and concatenating them back. The first few steps of this process are as follows:

- 0
- 0, *1*
- $0, 1,$ *1, 0*
- $0, 1, 1, 0,$ *1, 0, 0, 1*
- $0, 1, 1, 0, 1, 0, 0, 1,$ *1, 0, 0, 1, 0, 1, 1, 0*

Notice that this corresponds directly to the procedure above.

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Sources. The problems are original, although they are based on relatively well-known mathematical results.