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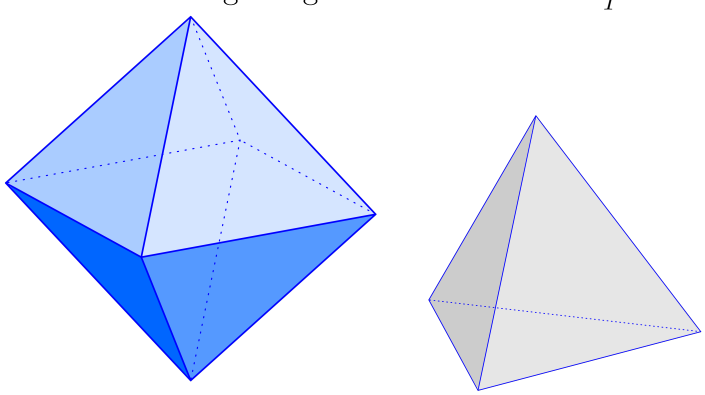
TEAM ROUND / 1 HOUR / 210 POINTS
November 16, 2013

WITH SOLUTIONS

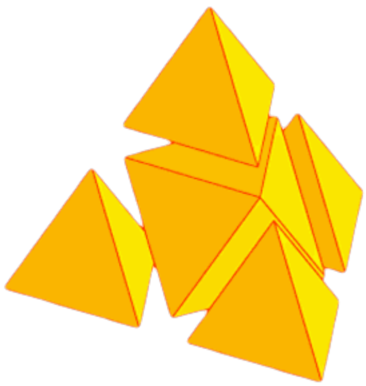
No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Octahedron vs tetrahedron)**.** Let O be the volume of a regular octahedron with edge length 1, and let T be the volume of a regular tetrahedron with edge length 1. Find the ratio $\frac{O}{T}$.



Answer. 4

Solution.



The key to the problem is to start with a tetrahedron of edge length 2. On each triangular face, connect the midpoints of the sides. These lines partition the tetrahedron into 4 tetrahedra of edge length 1 and an octahedron of edge length 1. We let

$$\begin{aligned} T_2 &= \text{volume of tetrahedron of edge length 2,} \\ T &= \text{volume of tetrahedron of edge length 1,} \\ O &= \text{volume of octahedron of edge length 2.} \end{aligned}$$

Then $T_2 = 2^3 \cdot T$, and $T_2 = 4T + O$. Hence, $O = 4T$, and $\frac{O}{T} = 4$.

Problem 2 (Nonstandard primes)**.** By a **binary string**, we mean a finite nonempty sequence of 0s and 1s, with no leading 0s unless the string consists only of 0. Listing strings by length, the first few examples are thus 0, 1, 10, 11, 101, ... We define **non-carry addition** (+) and **non-carry multiplication** (×) of binary strings by the usual grade-school algorithms for addition and multiplication **but systematically ignoring carries**. For example, $1 + 1 = 0$ with our definition, and

$$\begin{array}{r} 10101 \\ +\;\;1101 \\ \hline 11000 \end{array} \qquad \text{while} \qquad \begin{array}{r} 10101 \\ \times\qquad 1101 \\ \hline 10101 \\ 00000 \\ 10101 \\ 10101 \\ \hline 11101001 \end{array}$$

A **prime** is a binary string with more than one digit which cannot be written as a non-carry product except as $1 \times$ itself or itself $\times 1$. For example, 10 and 11 are prime, but 11101001 is not.

How many primes are there with exactly six digits?

Answer. 6

Solution. To each binary string $a_d a_{d-1} a_{d-2} \cdots a_1 a_0$, we associate the polynomial $a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$. Then the rules for addition and multiplication correspond precisely to the rules for addition and multiplication of polynomials, except that addition and multiplication is always done modulo 2. For example, $11 \times 11 = 101$, since modulo 2,

$$(x+1)^2 = x^2 + 2x + 1 = x^2 + 1.$$

Remember that $2 = 0$ when one works modulo 2.

Seen from this point of view, the problem is asking for the number of degree 5 polynomials that are irreducible when considered modulo 2. To begin with, there are 2^5 polynomials of degree 5 in total, namely

$$P(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0,$$

where each a_i is either 0 or 1. (If you prefer to think in terms of strings, these are the strings $1a_4 a_3 a_2 a_1 a_0$.)

As a warm up, let's determine the irreducibles of degree 2. There are only four degree 2 polynomials considered mod 2, namely $x^2 + x + 1$, $x^2 + x$, $x^2 + 1$, and x^2. The second and fourth are clearly divisible by x. And $x^2 + 1$ is divisible by $x + 1$; one easy way to see this is to use the **remainder theorem**. According to that result, the remainder when you divide a polynomial by $x + 1$ is obtained by plugging in -1. Working mod 2, we have $-1 = +1$, and

$$1^2 + 1 = 2 = 0.$$

So $x^2 + 1$ is not irreducible. This leaves only $x^2 + x + 1$. This clearly leaves a remainder of 1 when divided by x, and if we plug in 1, we get $1^2 + 1 + 1 = 1$, so $x^2 + x + 1$ is not divisible by $x + 1$. Since we've ruled out degree 1 factors of $x^2 + x + 1$, we see that $x^2 + x + 1$ is irreducible, and in fact the only irreducible of degree 2.

OK, let's try degree 3. Ruling out polynomials that are divisible by x, we are left with 4 candidates: $x^3 + x + 1$, $x^3 + x^2 + 1$, $x^3 + x + x^2 + 1$, and $x^3 + 1$. The final two are divisible by $x + 1$, by the remainder theorem again, while the first two are not. But if a polynomial of degree 3 factors, then one of the factors **must be** linear. So $x^3 + x + 1$ and $x^3 + x^2 + 1$ are irreducible, and are the only irreducibles of degree 3.

If a polynomial of degree 5 factors, then it has a factor of degree 1 or 2.

We rule out degree 1 factors as above and this leaves us with eight candidates:

$$x^5+x^4+x^2+x+1, \quad x^5+x^4+x^3+x+1, \quad x^5+x^4+x^3+x^2+1, \quad x^5+x^3+x^2+x+1,$$
$$\text{and} \quad x^5+x^4+1, \quad x^5+x^3+1, \quad x^5+x^2+1, \quad x^5+x+1.$$

If a degree 5 polynomial has no degree 1 factor, then the only way it can fail to be irreducible is if it has both a degree 2 irreducible factor and a degree 3 irreducible factor. We determined all degree 2 and 3 irreducibles above. So the only polynomials we have to cross of our list of eight are

$$(x^2+x+1)(x^3+x+1) = x^5+x^4+1$$

and

$$(x^2+x+1)(x^3+x^2+1) = x^5+x+1.$$

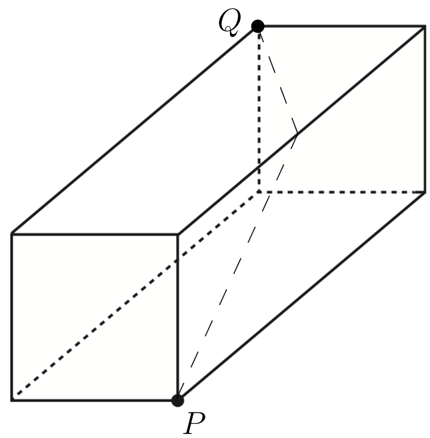
So we are left with **six** irreducibles of degree 5.

An advanced aside: Early in his spectacular career, Gauss came up with an exact formula for the number of polynomials of degree n that are irreducible modulo 2. When n is prime, his formula takes a particularly simple form, and predicts that the number of these polynomials is exactly

$$\frac{2^n-2}{n}.$$

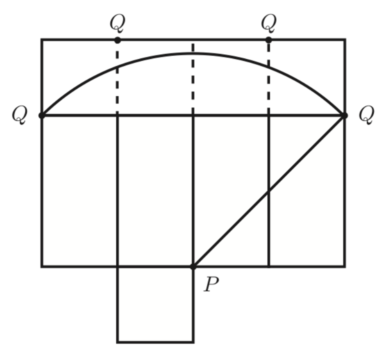
When $n=5$, Gauss's formula predicts $\frac{2^5-2}{5} = \frac{30}{5} = 6$ such polynomials, in agreement with our determination above.

Problem 3 (More rectangular boxing)**.** You may recall that on problem #15, you found that the distance from P to Q on the surface of a $1\times 1\times 2$ rectangular box is $\sqrt{8}$. (The dashed lines in the figure below show one path that achieves this minimum.) Surprisingly, Q is **not** the point on the surface of the box which is farthest from P. Find the distance from P to the point that is farthest from P.



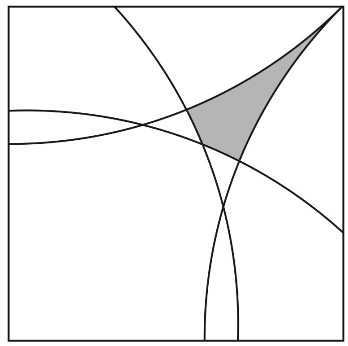
Answer. $\dfrac{\sqrt{130}}{4}$.

Solution. To get a feel for what's going on here, let's understand why the shortest path from P to Q has length $\sqrt{8}$. Draw a net, and a Euclidean circle of radius $\sqrt{8}$ centered at P on the net:



(You should check that the circular arcs of radius $\sqrt{8}$ centered at other representatives of P determine points that are inside the circular arc shown, and so do not lie on the circle of radius $\sqrt{8}$.)

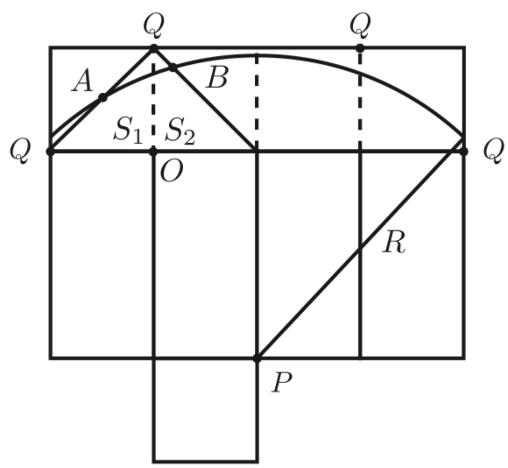
When we fold this net onto the box, notice that all of the circular arcs will lie on the back of the box; i.e., in the 1 by 1 square with Q in a corner. Here is that square:



The solid lines bounding the shaded region form the circle of radius $\sqrt{8}$, and the shaded region is the part outside of the circle. All the rest of the box is inside the circle.

In particular — here's the unintuitive part — there are points **outside** this circle. In other words, Q is **not** the farthest point from P in the surface metric.

Now increase the radius R. The point farthest from P will be the point at which the circle of radius R centered at P collapses to a single point. By symmetry, that point will lie on the diagonal of the 1 by 1 square starting at Q. So we need to find R so that the points A and B shown in the diagram correspond to the same point on the surface of the box.



To do that, let O be the origin, so that $P = (1, -2)$. Notice that, for points in S_1, $-1 < x < 0$ and $0 < y < 1$, while for points in S_2, $0 < x < 1$ and $0 < y < 1$. Also notice that (x, y) in S_2 corresponds to $(-y, x)$ in S_1 when the box is folded. Finally, recall that we want a point on the diagonal $y = 1 - x$. So we need to find the x so that $(x, 1 - x)$ and $(x - 1, x)$ are equidistant from P:

$$\sqrt{(x-1)^2 + (1-x+2)^2} = \sqrt{(x-2)^2 + (x+2)^2}.$$

Solving, $x = \frac{1}{4}$ and $y = 1 - x = \frac{3}{4}$. So the distance to the farthest point is

$$\sqrt{\left(\frac{1}{4} - 1\right)^2 + \left(\frac{3}{4} - (-2)\right)^2} = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{11}{4}\right)^2} = \frac{\sqrt{130}}{4}.$$

Authors. Problems and solutions were written by Mo Hendon, Paul Pollack, and Joe Tenini.

Sources. The pictures of the octahedron and tetrahedron in Problem 1 are from Wikipedia; the decomposition of the tetrahedron shown in the solution is from MatematicasVisuales:

`http://www.matematicasvisuales.com/english/index.html`