Sponsored by: UGA Math Department and UGA Math Club

TEAM ROUND / 1 HOUR / 210 POINTS
October 24, 2015

WITH SOLUTIONS

Problem 1 (Mathematics for fun and profit)**.** Here's how the Big Bucks Lottery works. When you buy a lottery ticket for \$1, you get to choose 3 different numbers from $\{1, 2, 3, 4, 5, 6, 7\}$. Once you've bought all the tickets you want, the lottery company randomly chooses 3 distinct numbers from $\{1, 2, 3, 4, 5, 6, 7\}$. For each of your tickets that matches all three numbers (jackpot!) you win \$10. For each of your tickets that matches exactly 2 numbers, you win \$3.

(a) (35 points) If you buy exactly one of every possible ticket, what will your profit be?

(b) (35 points) What is the smallest number of tickets you can buy and still be guaranteed to make a (positive!) profit?

Answer. (a) \$11 (b) 7 (tickets)

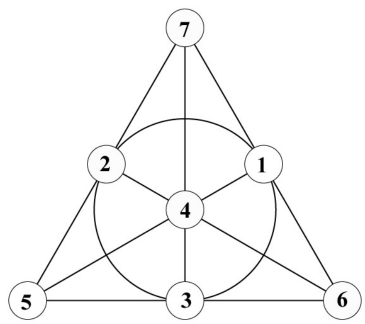
Solution. (a) There are $\binom{7}{3} = 35$ different tickets, so it will cost you \$35 to buy one of each. You'll have one jackpot (\$10), and for each 2 of the 3 winning numbers you'll have 4 more tickets matching that pair (e.g., if 123 wins, you'll also have 124, 125, 126, and 127). So your profit is

$$10 + 12 \cdot 3 - 35 = 11$$

dollars.

(b) It's certainly possible to guarantee a profit with less tickets — in fact, if you buy *one* less ticket, you might (worst case), miss the jackpot, but you still get 12 winning pairs, for a profit of \$2.

But, surprisingly, it's possible to guarantee a profit with as few as 7 tickets. To see this, we'll take advantage of a little bit of finite projective geometry. The *Fano plane* is a geometry with 7 points, which we'll call 1, 2, 3, 4, 5, 6, and 7. It's often represented with this diagram:



There are only seven points in the geometry — the extra curves and lines are there to indicate which points are to be considered collinear. Here are the lines in this geometry:

$$\{5,2,7\}, \quad \{7,1,6\}, \quad \{5,3,6\}, \quad \{2,4,6\}, \quad \{1,4,5\}, \quad \{7,4,3\}, \quad \{2,1,3\}.$$

You may object that the last of these, $\{2,1,3\}$, doesn't look like a line, but they all satisfy the most important property of a line: two points determine a unique line.

How can you use this to win the lottery? Simply buy the seven tickets corresponding to the seven lines above. If one of them wins the jackpot, you've paid \$7 to win \$10, a profit of \$3. If none of them wins the jackpot, then the three winning numbers are not collinear, so the 3 pairs determine 3 distinct lines — and you have the tickets corresponding to those lines! So you've paid \$7 to win $3 \cdot \$3 = \$9$, a profit of \$2.

To see that you can't guarantee a profit with six or fewer tickets, notice that if you buy 6 tickets, you've covered $6 \cdot \binom{3}{2} = 18$ pairs or less (less if, for example, you buy 123 and 124). Since there are $\binom{7}{2} = 21$ pairs altogether, if the winning ticket contains one of the (at least) 3 uncovered pairs, you win neither the jackpot nor that pair, so you can win at most two pairs. Your winnings are then at most \$6, which at best offsets your \$6 expense. So at best you come out even. You should check that if you buy less than six tickets, it's possible that none of your tickets win anything.

If you're thinking that this type of thing never happens in a real lottery, think again. In his recent book *How Not to Be Wrong: The Power of Mathematical Thinking*, Jordan Ellenberg describes the (now defunct) Massachusetts WinFall lottery game. It was more complicated than this example — a ticket consisted of 6 out of 46 numbers — but it too offered a disproportionately high payoff for tickets that matched some but not all numbers. Several groups made millions of dollars exploiting this.

If you're thinking the state of Massachusetts was not too bright for running such a lottery, think again — they pocketed hundreds of millions of dollars. Think about that next time you consider buying a lottery ticket!

Problem 2 (A binary word problem)**.** The *Thue–Morse sequence* $t_0, t_1, t_2, \ldots$ is a sequence of 0s and 1s defined by the rule

$$t_n = \begin{cases} 0 & \text{if } n \text{ has an even number of 1s in its binary expansion,} \\ 1 & \text{otherwise.} \end{cases}$$

For example, $t_0 = 0$ and $t_{13} = 1$. If the terms of the sequence are concatenated, one obtains an infinite "word" in the letters 0 and 1 which begins

$$0110100110010110\ldots,$$

where we take the starting "letter" to be $t_0 = 0$. How many occurrences of the string 11 are there in the initial segment

$$t_0 t_1 \ldots t_{2014} t_{2015}?$$

In other words, for how many integers $0 \leq n < 2015$ is $t_n = t_{n+1} = 1$?

Answer. 336

Solution. The infinite string $t_0 t_1 t_2 \ldots$ may be generated by the following process: First, begin with 0, which is t_0. Having constructed the string $t_0 \ldots t_{2^j-1}$ (for some $j \geq 0$), we form the string

$$t_0 t_1 \ldots t_{2^{j+1}-1}$$

by tacking on to the end of our current string its bitwise negation. (To see why this makes sense, notice that the binary expansions of the numbers $2^j, 2^j+1, \ldots, 2^{j+1}-1$ are formed by adding a leading 1 to the expansions of $0, 1, \ldots, 2^j - 1$.) To illustrate the process, the first several steps are

$$\begin{aligned}
&0\\
&01\\
&0110\\
&01101001\\
&0110100110010110\\
&01101001100101101001011001101001.
\end{aligned}$$

Let A_j and B_j be the number of occurrences of the strings 00 and 11 within $t_0 t_1 \cdots t_{2^j-1}$. From the above process, we can read off that

$$A_{j+1} = A_j + B_j,$$

since the only time we see a 00 comes from either the initial string of 2^j terms, or from seeing a 11 in its bitwise negation. For B_j, we have the more complicated relation

$$B_{j+1} = A_j + B_j + \begin{cases} 1 & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

Notice that if j is odd, when we go from the initial string of 2^j terms to the string of 2^{j+1} terms, the ending 1 in the first string is connected with the opening 1 in the bitwise-negated string; this explains the extra $+1$ we see above for those j.

Starting with $A_0 = B_0 = 0$, we find successively $A_1 = B_1 = 0$, then $A_2 = 0$ and $B_2 = 1$, then $A_3 = B_3 = 1$, then $A_4 = 2$ and $B_4 = 3$, then $A_5 = B_5 = 5$, then $A_6 = 10$ and $B_6 = 11$, then $A_7 = B_7 = 21$, then $A_8 = 42$ and $B_8 = 43$, then $A_9 = B_9 = 85$, then $A_{10} = 170$ and $B_{10} = 171$, then $A_{11} = B_{11} = 341$.

Now B_{11} is the number of solutions to $t_n = t_{n+1}$ with $0 \leq n < 2047$. From this count, we need to subtract off the number of solutions for $2015 \leq n < 2047$. Since the binary expansion of 2047 consists of eleven 1s, it follows that t_{2047-n} is the bitwise negation of t_n, for all $0 \leq n \leq 2047$. So letting

$k = 2047 - n$,

$$\begin{aligned}\#\{2015 \le n < 2047 : t_n = t_{n+1} = 1\} \\ = \#\{0 < k \le 32 : t_{k-1} = t_k = 0\} \\ = \#\{0 \le m \le 31 : t_m = t_{m+1} = 0\}.\end{aligned}$$

The final expression here is *almost* our A_5: A_5 is the number of solutions to $t_m = t_{m+1} = 0$ for $0 \le m < 31$. We only see a difference if $t_{31} = t_{32} = 0$. But $t_{31} = 1$. So in fact, the last displayed quantity *is* $A_5 = 5$. Hence, the final answer is $341 - 5 = 336$.

Problem 3 (Be careful or you'll lose a digit!)**.** Dan D. Man (the D stands for "Digit") tabulates the leading decimal digits of each of the 2015 numbers $3^0, 3^1, \ldots, 3^{2014}$. He observes that 3^{2014} has leading digit 8 and that the digit 9 appears 93 times as the leading digit. If A is the number of times that 1 appears the leading digit, and B the number of times that 2 appears, find $A + B$.

Answer. 961

Solution. Consider the sequence ℓ_n giving the leading digits of 3^n, for $n = 0, 1, 2, \ldots$. The consecutive terms of the sequence can be broken into nonoverlapping groups of the following types:

$$\begin{aligned}a = \{1, 3, 9\}, \quad b = \{1, 3\}, \quad c = \{1, 4\}, \quad d = \{1, 5\}, \\ e = \{2, 6\}, \quad f = \{2, 7\}, \quad g = \{2, 8\}.\end{aligned}$$

Since 3^{2014} has leading digit 8, we know that the sequence $\ell_0, \ldots, \ell_{2014}$ ends with the appearance of a group of type g.

Let N_a be the number of groups of type a that occur in $\ell_0, \ldots, \ell_{2014}$, and similarly for N_b thru N_g. Clearly,

$$3N_a + 2(N_b + N_c + N_d + N_e + N_f + N_g) = 2015.$$

Since 9 appears 93 times as the leading digit, we know $N_a = 93$; subtracting N_a from the preceding equation gives

$$2(N_a + N_b + N_c + N_d + N_e + N_f + N_g) = 1922.$$

We seek $N_a + N_b + N_c + N_d + N_e + N_f + N_g$, which is $\frac{1}{2} \cdot 1922 = 961$.

Authors. Problems and solutions were written by Mo Hendon and Paul Pollack.

Sources. The image of the Fano plane was "borrowed" from an MAA article on projective geometry: `http://www.maa.org/community/maa-columns/past-columns-card-colm/projective-geometry-the-fano-plane`. Problem #3 was inspired by #25 on AMC 12B, 2004.