University of Georgia High School Math Tournament

2005

Sponsored by: UGA Math Department and UGA Math Club

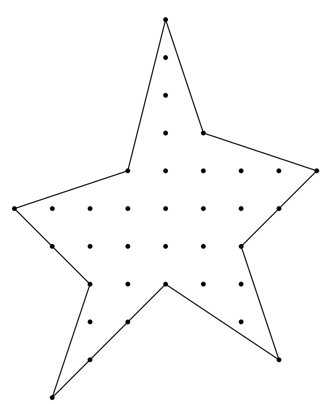
Written test, 25 problems / 90 minutes

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

Problem 1. The star below is drawn on the standard rectangular grid on which every square has size 1 by 1. What is the area of the star?



(A) 26.5 (B) 27 (C) 27.5 (D)$^\heartsuit$ 28 (E) None of the above.

Solution. *First way.* Just cut into triangles and count their areas separately.

Second way. Use *Pick's formula:* the area of a polygon with vertices that have integral coordinates is given by the formula

$$A = I + \frac{B}{2} - 1,$$

where I is the number of integral points in the interior and B is the number of integral points on the boundary, including the vertices. We get:

$$A = 22 + \frac{14}{2} - 1 = 28.$$

Problem 2. The Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

have the property that every number of this sequence, starting with the third, is the sum of two previous numbers (for example, $8 = 5 + 3$). What is the greatest common divisor of the 2004th and 2005th Fibonacci numbers?

(A)$^\heartsuit$ 1 (B) 2 (C) 3 (D) 5 (E) None of the above

Solution. Let (a, b) denote the greatest common divisor of a and b. We have:

$$(F_{2005}, F_{2004}) = (F_{2004} + F_{2003}, F_{2004}) = (F_{2003}, F_{2004})$$

Therefore,

$$(F_{2005}, F_{2004}) = (F_{2004}, F_{2003}) = (F_{2003}, F_{2002}) = \cdots = (F_2, F_1) = (1, 1) = 1$$

Problem 3. As has been recently discovered, there are three types of amoebas on Mars: types A, B and C. It is also known that if two amoebas of different types fuse together, they form an amoeba of the third type (for example, a type B amoeba and a type C amoeba fuse to form a type A amoeba). After a certain time, there remains only one amoeba. If originally there were 20 type A, 21 type B and 22 type C amoebas, what type is the last remaining amoeba?

(A) A (B)$^\heartsuit$ B (C) C (D) The answer is not uniquely determined. (E) More than one amoeba must remain at the end (for example, if two type A amoebas remain, no more fusion can occur) if one begins with the specified numbers.

Solution. If all of the type A amoeba fuses with all but one of the type B amoebas, there remains one type B amoeba and 40 type C amoebas. Now, the type C amoebas can alternatingly fuse with type B and type A amoebas until only one amoeba remains, of type B. The arguments below demonstrate that the answer is uniquely determined.

Second proof. However the amoebas fuse, the parity (even/oddness) of the number of type A amoebas *plus* the number of type B amoebas remains constant. We can argue similarly for types B and C, and C and A. Thus, in the end, the remaining amoeba must be of type B. This argument also demonstrates uniqueness.

Third proof. Consider the Klein-four abelian group $\mathcal{V}_4 = \{0, a, b, c\}$ with the following addition rules:

$$\begin{array}{ccc} 0+a=a & 0+b=b & 0+c=c \\ a+a=0 & b+b=0 & c+c=0 \\ a+b=c & b+c=a & c+a=b \end{array}$$

We notice that because of this last property, we need simply to simplify $20a+21b+22c$ in $\mathcal{V}_4$; but $20a = 20b = 22c = 0$, so $20a+21b+22c = b$.

Problem 4. $x + \dfrac{1}{x} = 3$. What is $x^4 + \dfrac{1}{x^4}$?

(A) 7 (B) 9 (C)$^\heartsuit$ 47 (D) 81 (E) None of the above

Solution.

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = 3^2 - 2 = 7$$
$$x^4 + \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = 7^2 - 2 = 47$$

Problem 5. Let $f(n) = n^3$ and define the function $g(n)$ by the formula

$$g(n) = f(n+1) - f(n)$$

What is the average of the 10 numbers $g(0), g(1), \dots, g(9)$?

(A) 64 (B) 81 (C) 95 (D) 105 (E)$^\heartsuit$ None of the above

Solution.

$$\begin{aligned}&g(0) + g(1) + \cdots + g(9) = \\ &(f(1) - f(0)) + (f(2) - f(1)) + \cdots + (f(10) - f(9)) = \\ &f(10) - f(0) = 10^3 - 0^3 = 1000,\end{aligned}$$

so the average is $1000/10 = 100$.

Problem 6. Let $P(x)$ be a polynomial such that

$$\big(P(x)\big)^2 = 1 - 2\,x + 5\,x^2 - 4\,x^3 + 4\,x^4$$

What is $|P(1)|$?

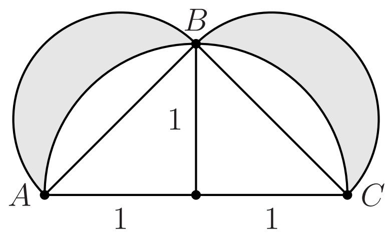
(A) 0 (B) 1 (C)$^\heartsuit$ 2 (D) 3 (E) 4

Solution. Do not try to find $P(x)$! Instead, note that

$$\big(P(1)\big)^2 = 1 - 2 + 5 - 4 + 4 = 4, \quad \text{so} \quad |P(1)| = \sqrt{4} = 2.$$

(However, if you do try to compute $P(x)$, you should find that $P(x) = \pm(2x^2 - x + 1)$.)

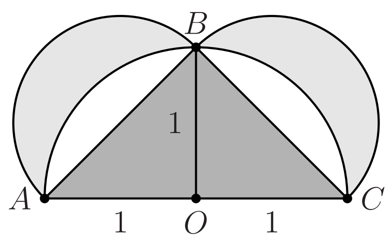
Problem 7. On the picture below, the diameters of the three semicircles are sides of the right triangle ABC. The angles A and C are 45 degrees each.



Find the shaded area.

(A)$^\heartsuit$ 1 (B) $\dfrac{\pi}{2} - 1$ (C) $\dfrac{\pi}{4}$ (D) $\dfrac{\pi}{2}$ (E) $\dfrac{3\pi}{2} - 1$

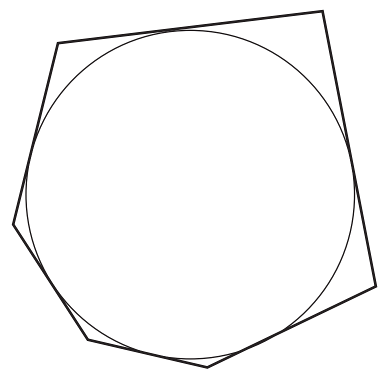
Solution.



The lengths of the diameters AC and AB are 2 and $\sqrt{2}$ respectively. Hence, the area of the half-circle with diameter AC equals twice the area of the half-circle with diameter AB.

The left half-circle and the left quarter-circle have the white region in common. The same is true for the right side of the picture. Therefore, the light-shaded area is the same as the dark-shaded area, and the latter obviously is 1.

Problem 8. A circumscribed hexagon has sides 2, 3, 5, 7, 9 and x in clockwise order. What is x?



(A) 2 (B) 4 (C)$^{\heartsuit}$ 6 (D) 8 (E) None of the above

Solution. We claim that the sum of the 1st, 3rd and 5th sides equals the sum of the 2nd, 4th and 6th. Indeed, at every vertex of the hexagon there are two equal intervals from this vertex to the points of tangency, one of them appears in the first sum and the other in the second sum.

So, $2+5+9=3+7+x$ and $x=6$.

Problem 9. What is the last digit of

$$7^{7^{7^{7}}}\ ?$$

(A) 1 (B)$^{\heartsuit}$ 3 (C) 5 (D) 7 (E) 9

Solution. The last digits of powers $7^0, 7^1, 7^2, \ldots$ are $1, 7, 9, 3, 1, 7, 9, 3, \ldots$. Therefore, to find it, we only need to determine

$$7^{7^{7}} \mod 4$$

But $7^0, 7^1, 7^2, \ldots$ modulo 4 are $1, 3, 1, 3, \ldots$. Since $N = 7^{7^7}$ is odd, $7^N = 7^3 \mod 10 = 3 \mod 10$.

Problem 10. There are 8 empty chairs in a row. In how many ways can one seat 3 people so that no two of them sit next to one another?

(A)$^\heartsuit$ 20 (B) 24 (C) 56 (D) 720 (E)$^\heartsuit$ None of the above

Solution. There will be 5 empty chairs. Between them and at the ends (so, in 6 possible positions) one can sit one person. So, there are $\binom{6}{3} = 20$ possibilities.

P.S. The way the problem is written, $6 \times 20 = 120$ is also a correct answer, if one takes into account the $6 = 3!$ different orders in which 3 people can sit.

2 Medium Problems

Problem 11. What is $(1+i)^{11}$?

(A) $16 + 16i$ (B) $-16 + 16i$ (C) $32 + 32i$ (D)$^\heartsuit$ $-32 + 32i$ (E) None of the above

Solution.

$$(1+i)^{11} = \left((1+i)^2\right)^5 (1+i) = (2i)^5(1+i) = 32i(1+i) = -32 + 32i$$

Problem 12. In a mythical country people have only two kinds of coins: 7 and 9 cents. What is the largest amount that *cannot* be made using these coins?

(A) 38 (B) 40 (C)$^\heartsuit$ 47 (D) 48 (E) None of the above

Solution. For two coins with relatively prime values a and b the minimal impossible amount is $ab-a-b = (a-1)(b-1)-1$, and this is called Sylvester's theorem. The proof is as follows. It works for any relatively prime a and b but we will only write it for 7 and 9 for simplicity.

Write all the nonnegative integers 0, 1, 2, etc., in rows, 7 in each row:

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ \ldots \end{array}$$

Since 0 is divisible by 9, it and all numbers below it can be obtained with our coins, so cross out all this column out. Next, cross 9 and all numbers below it, 18 and all numbers below it, etc. Now notice that the first 7 numbers: $0, 9, 18, \ldots, 9(7-1)$ are all in different columns because 7 and 9 are relatively prime. Therefore, the largest number not crossed out will be right above $9(7-1)$; in other words, it will be

$$9(7-1) - 7 = 9 \cdot 7 - 9 - 7 = ab - a - b.$$

Problem 13. The Fibonacci numbers $f_1, f_2, \ldots$ are $1, 1, 2, 3, 5, 8, 13, \ldots$ (every number is the sum of the previous two). Find

$$\sum_{n=1}^{\infty} \frac{f_n}{2^n}$$

(A) 1 (B)$^\heartsuit$ 2 (C) 3 (D) 4 (E) None of the above

Solution. Let the sum be S and set $f_0 = 0$. Then

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} f_n \left(\frac{1}{2}\right)^n \\
&= \frac{1}{2} + \sum_{n=2}^{\infty} f_n \left(\frac{1}{2}\right)^n \\
&= \frac{1}{2} + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) \left(\frac{1}{2}\right)^n \\
&= \frac{1}{2} + \frac{1}{2}(S-0) + \frac{1}{4}S = \frac{1}{2} + \frac{3}{4}S
\end{aligned}$$

Solving for S gives $S = 2$.

Second proof. The generating function for the Fibonacci sequence is

$$\sum f_n x^n = \frac{x}{1-x-x^2}$$

Plugging in $x = 1/2$ we get

$$\frac{1/2}{1-1/2-1/4} = 2$$

Problem 14. How many times during a day (24 hours) do the hour hand and the minute hand on the clock point in opposite directions?

(A) 2 (B)$^{\heartsuit}$ 22 (C) 24 (D) 20 (E) None of the above

Solution. We will find the number for the period of 12 hours and then multiply the answer by two to get the answer for the 24-hour day.

Let $0 \le x < 12$ represent the hours (it need not be integral, f.e. 1 hour 25 minutes means that $x = 1 + 25/60$). Then the minute hand points at $y = 12x \mod 12$. Our condition says that the following condition must be satisfied:

$$\begin{aligned}
&y = x + 6 \mod 12, \quad \text{i.e., } 12x = x + 6 \mod 12 \\
&11x = 6 \mod 12
\end{aligned}$$

This equation has 11 solutions: $6/11$, $(6+12)/11$, $\ldots$, $(6+10\cdot 12)/11$. For the 24-hour day, we obtain $2\times 11 = 22$ solutions.

Alternative solution. Evident solution at 6:00. We get another solution either $\frac{12}{11}$ before or after (since $\frac{12}{11} = 1 + \frac{1}{12}\cdot\frac{12}{11}$); and $\frac{12}{11}$ after that, etc.; 11 solutions in all.

Problem 15. Take a random irrational number such as $\pi = 3.1415926535\cdots$, and add up the first k digits after the decimal point. For example, with $k = 5$, $1+4+1+5+9 = 20$. What is the probability that, for some k, this sum is 2005? The answer is closest to:

(A) 0 (B) 1/10 (C) 1/9 (D)$^\heartsuit$ 1/5 (E) 2/9

Solution. Every time you add a nonzero digit, it is one of the digits 1–9. As a result, the average "step" of the partial sums is 5. Thus, the density of the numbers which occur as partial sums is 1/5.

Notice that we must ignore the digit 0, as otherwise we would be double-counting in the density argument.

Problem 16. Which of the following numbers can not be written as a difference of two perfect squares?

(A)$^\heartsuit$ 20,000,002 (B) 20,000,003 (C) 20,000,004 (D) None of these (E) More than one of these

Solution. A number that is equal to 2 mod 4 cannot be written as a difference of two perfect squares. Indeed, every perfect square equals 0 or 1 mod 4, so the differences can be 0 and ± 1 mod 4.

For an odd number $n = 2k+1$, we have $2k+1 = (k+1)^2 - k^2$.

For a multiple of 4, $n = 4k$, we have $4k = (k+1)^2 - (k-1)^2$.

The three numbers above are equal to 2, 3 and 0 respectively mod 4; so only 20000002 is not a difference of squares.

Second Proof. Notice that $n = x^2 - y^2 = (x+y)(x-y)$ and that the numbers $x+y$ and $x-y$ have the same parity, i.e., both odd or both even. Vice versa, if $n = pq$ and p, q have the same parity then $p = x+y$ and $q = x-y$, if we take $x = (p+q)/2$ and $y = (p-q)/2$.

Any odd number can be written as a product of two odd numbers, f.e. $n = 1 \cdot n$. Any number divisible by 4 can be written as a product of two even numbers, f.e. $4k = 2 \cdot 2k$. A number that equals 2 mod 4 obviously cannot be written as a product of two odd numbers, or of two even numbers.

Problem 17. Compute

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 7 \cdot 7!$$

(A) 40,019 (B) 40,119 (C) 40,329 (D) 3,528,799 (E)$^\heartsuit$ None of the above

Solution. Notice that

$$(n+1)! - n! = (n+1)n! - n! = n \cdot n!$$

Therefore, the sum above is

$$(2! - 1!) + (3! - 2!) + \cdots (8! - 7!) = 8! - 1! = 40320 - 1 = 40319$$

since $2!, 3!, \ldots, 7!$ cancel. (Sums like these are called *telescoping sums*.)

Problem 18. How many numbers between 1 and 2004 are relatively prime to 2005 (i.e., the two numbers have greatest common divisor 1)?

(A) 800 (B) 805 (C) 1604 (D) 2000 (E)$^\heartsuit$ None of the above

Solution. The function in question is called Euler's function $\phi(n)$ and its main properties (prove them!) are

1. $\phi(ab) = \phi(a)\phi(b)$ if a and b are relatively prime,
2. $\phi(p) = p - 1$ if p is a prime number
3. (we don't need this one) $\phi(p^k) = p^k - p^{k-1}$ if p is a prime number.

Now, $2005 = 5 \cdot 401$, and both 5 and 401 are prime. So,

$$\phi(2005) = \phi(5)\phi(401) = 4 \cdot 400 = 1600$$

Problem 19. When you divide the polynomial $x^{2005} + x + 2$ by $x^2 - 1$, what will be the remainder?

(A) $2x - 2$ (B) $-2x - 2$ (C) $-2x - 2$ (D) 2 (E)$^\heartsuit$ None of the above

Solution. $x^{2005} - x^{2003}$ is divisible by $x^2 - 1$, and so is

$$(x^{2005} - x^{2003}) + (x^{2003} - x^{2001}) + \cdots (x^3 - x) = x^{2005} - x$$

Hence, remainder will be

$$x^{2005} + x + 2 - (x^{2005} - x) = 2x + 2$$

Alternative solution.

$$x^{2005} + x + 2 = q(x)(x^2 - 1) + (ax + b)$$

for some $a, b \in \mathbb{R}$. Substituting $x = -1$ and $x = 1$,

$$0 = -a + b, \qquad 4 = a + b$$

Solving, we obtain $ax + b = 2x + 2$.

Problem 20. Two ships move with constant speeds and directions. At noon, 2 p.m., and 3 p.m. respectively the distance between them was respectively 5, 7, and 2 miles. What was the distance between them at 1 p.m.?

(A) 6 (B) 10 (C) $\sqrt{50}$ (D)$^\heartsuit$ $\sqrt{56}$ (E)$^\heartsuit$ None of the above

Solution. Note the following: since x- and y- coordinates of the ships are linear functions of time t, the *square* of the distance between them — by the distance formula — is a quadratic function of t:

$$d(t)^2 = at^2 + bt + c$$

Substituting $t = 0, 2, 3$, we obtain

$$\begin{aligned} t = 0 &\implies c = 25 \\ t = 2 &\implies 4a + 2b + 25 = 49 \\ t = 3 &\implies 9a + 3b + 25 = 4 \end{aligned}$$

Solving, we obtain $a = -19$ and $b = 50$. Therefore, $d(1)^2 = 56$ and $d = \sqrt{56}$.

P.S. It was pointed out to us that $a = -19$ is impossible, since this says that for $t \gg 0$ the distance is going to be $-\infty$. Therefore, we also accepted "None of the above" as correct answer.

3 Hard Problems

Problem 21. Five points are placed on a sphere of radius 1 such that the sum of the squares of the pairwise distances between them are maximized. (There are 10 terms in this sum.) What is this sum?

(A) 20 (B) 21 (C) 24 (D) 30 (E)$^\heartsuit$ None of the above

Solution. Place the points at the ends of unit vectors $v_1, \ldots, v_5$. We need to maximize

$$|v_1 - v_2|^2 + \cdots + |v_4 - v_5|^2 = \sum_{i<j} |v_i - v_j|^2 .$$

(Note that there are $\binom{5}{2} = 10$ terms in the sum here.) But remember that $|v \pm w|^2 = |v|^2 \pm 2v \cdot w + |w|^2$, so

$$|v_1+\cdots+v_5|^2+|v_1-v_2|^2+\cdots+|v_4-v_5|^2 = 5\big(|v_1|^2+|v_2|^2+\cdots+|v_5|^2\big) = 5\cdot 5 = 25.$$

Thus, our sum is maximized when $|v_1 + \cdots + v_5|^2$ is minimized. This occurs when $v_1+\cdots+v_5 = 0$, and so our sum has maximum value 25. (Note that we can arrange the points at the vertices of a regular pentagon, or, alternatively, with one point at the north pole, one at the south pole, and the remaining three at the vertices of an equilateral triangle on the equator.)

Problem 22. What is the 100th smallest positive integer that can be written as the sum of **distinct** powers of 3, i.e., $1, 3, 9, 27, \ldots$?

(A) 969 (B) 973 (C) 974 (D)$^\heartsuit$ 981 (E) None of the above

Solution. In base three, there are just the numbers that can be written using only 0 and 1. The 100th number can be found as follows:

$$100 = 64 + 32 + 4 = (1100100)_2, \quad \text{and} \quad (1100100)_3 = 729 + 243 + 9 = 981$$

Problem 23. Find

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ.$$

(A)$^\heartsuit$ 1/16 (B) $\sqrt{2}/24$ (C) 1/12 (D) 1/8 (E) None of the above

Solution. First of all, $\cos 60^\circ = 1/2$. Multiply the rest by $\sin 20^\circ$ and use the formula $\sin x \cos x = (\sin 2x)/2$. We get:

$$\begin{aligned}\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ &= \sin 40^\circ \cos 40^\circ \cos 80^\circ/2 = \sin 80^\circ \cos 80^\circ/4 \\ &= \sin 160^\circ/8 = \sin 20^\circ/8\end{aligned}$$

Hence, the product is

$$(1/8)(1/2) = 1/16$$

Problem 24. Ten people are standing in line to buy movie tickets that cost \$5. Five of them have only \$5 bills and the other five have only \$10 bills. The cashier has absolutely no change, so a person with a \$5 bill must come before a person with a \$10 bill or else the sale will not happen.

What is the probability that they will happen to stand in line just right so that all ten will be able to buy tickets?

(A) 1/10 (B)$^\heartsuit$ 1/6 (C) 1/5 (D) 1/2 (E) None of the above

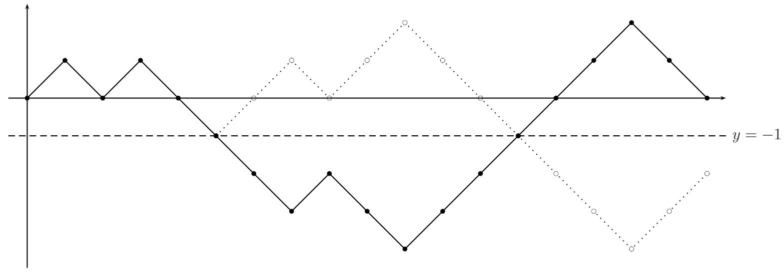
Solution. The number of possible ways to put $n+n$ people the right way is called the n-th *Catalan number* C_n. Richard Stanley gives 130 interpretations for Catalan numbers in his "Catalan addendum" at `http://math.mit.edu/~rstan/ec/catadd.pdf`. The formula for the n-th number is

$$C_n = \frac{\binom{2n}{n}}{n+1},$$

and the total number of ways to put $n+n$ people with \$5 and \$10 bills in line is $\binom{2n}{n}$. So the probability of getting it right is $1/(n+1)$, which for $n = 5$ gives $1/6$.

To prove that the formula for the n-th Catalan number is given by the above formula, consider the following interpretation: to each person, associate either $+1$ or -1. People with \$5 bills receive a $+1$ and people with \$10 bills receive a -1; we wish to arrange n $(+1)$s and n (-1)s so that all partial sums are nonnegative. In other words, we wish to calculate the number of lattice paths beginning at $(0,0)$ and ending at $(2n,0)$ with steps of ± 1 (such a path is shown below using solid lines) that never dips below the line $y = 0$. Clearly, this is simply the total number of lattice paths, $\binom{2n}{n}$, minus the number of paths that touch or cross the line $y = -1$. It remains to count the number of such "bad" paths.

We claim that there are exactly $\binom{2n}{n-1}$ bad paths; to show this, we demonstrate a correspondence between them and lattice paths of the ± 1 sort that end at $(2n, -2)$ (the number of which is $\binom{2n}{n-1}$ because one must choose $n-1$ positions for -1s among $2n$ total ± 1s). Consider the first time such a path (shown below using dotted lines) touches the line $y = -1$ and right of it, reflect across the line $y = -1$; you obtain a one-to-one correspondence with paths from $(0,0)$ to $(2n, 0)$ that touch or cross the line $y = -1$. Thus, the n-th Catalan number is $\binom{2n}{n} - \binom{2n}{n-1} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}$.



Problem 25. Let $\phi(n)$ denote *Euler's function*, the number of integers $1 \le i < n$ that are relatively prime to n. (For example, $\phi(9) = 6$ and $\phi(10) = 4$.) What is the last digit of the sum

$$\sum_{d|2005} \phi(d)$$

which goes over all (positive integral) divisors d of 2005 including 1 and 2005.

(A) 0 or 1 (B) 2 or 3 (C)$^\heartsuit$ 4 or 5 (D) 6 or 7 (E) 8 or 9

Solution. Using the Gauss identity $\sum_{d|n} \phi(d) = n$, the answer is obviously 5. Here is how to prove the Gauss identity. If $n = p^k$ is a power of a prime then

$$\sum_{d|n} \phi(d) = \phi(p^k) + \phi(p^{k-1}) + \cdots + \phi(p) + \phi(1)$$
$$= (p^k - p^{k-1}) + (p^{k-1} - p^{k-2}) + \cdots + (p-1) + \phi(1) = p^k = n$$

One of the main properties of $\phi(n)$ is that it is multiplicative: $\phi(nm) = \phi(n)\phi(m)$ if n and m are relatively prime. Now, note that the function

$\psi(n) = \sum_{d|n} \phi(d)$ is also multiplicative. So, once we have proved the identity $\psi(n) = \phi(n)$ for powers of primes, we have also proved it for their products, i.e., for arbitrary positive integers.

Authors. Written by Valery Alexeev and Boris Alexeev ©2005, with assistance by Ted Shifrin and Mo Hendon. Some problems were taken from N.B. Alfutova, A.B. Ustinov "Algebra and number theory for mathematical schools" published by Moscow Center for Continuing Mathematical Education, 2002.