1785

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 2, 2010

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

Problem 1. What is the closest integer to

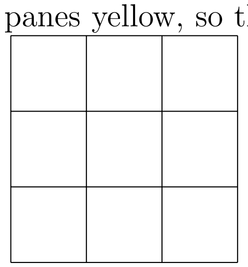
$$\frac{248163264}{123456}?$$

(A) 2 (B) 20 (C) 201 (D)$^\heartsuit$ 2010 (E) 201012

Solution. This problem is just intended as warmup. You can figure out the answer either by just estimating the order of magnitude of the quotient (for example, by counting digits), or by actually beginning long division. The approximate value of the quotient is 2010.135.

Remark. Let a_n be the concatenation of the first n powers of 2 beginning with 2 itself, b_n be the concatenation of the first n positive integers, and c_n the greatest integer less than or equal to a_n/b_n. (For example, $a_6 = 248163264$, $b_6 = 123456$, and $c_6 = 2010$.) Then the answers above are c_3 through c_7. The sequence c_n appears as sequence A067097 in the Online Encyclopedia of Integer Sequences `http://www.research.att.com/~njas/sequences`.

Problem 2. A window has 9 panes in the form of a 3×3 grid, as in the picture. In how many ways can one color 6 of these panes yellow, so that the window looks the same from inside and outside the house?



(A) 6 (B) 9 (C)$^\heartsuit$ 10 (D) 36 (E) None of the above

Solution. Of course, we can first color all the panes yellow, and then color 3 panes white, so we will solve the easier problem, for 3 panes. The solution has to be mirror symmetric, if we reflect along the vertical middle line.

In the central column there could be either 3 or 1 white panes. For 3, there is only one possibility. For 1 white pane in the middle there are $3 \cdot 3 = 9$ possibilities: 3 for the middle one, and 3 for the white panel in the left column; then we have to color a pane in the similar position on the right. The total is $1 + 9 = 10$.

Problem 3. How many different ways are there to place seven rooks on a chessboard so that no two attack each other or occupy the same square?

Recall that a chessboard is an 8 by 8 grid. A rook attacks all the squares in the row *and* column that it occupies, a total of fifteen squares.

(A) 5040 (B) 40320 (C)$^\heartsuit$ 322560 (D) 362880 (E) None of the above

Solution. Consider such a position of seven rooks. The rooks must occupy seven different rows and seven different columns, so they may be completed in a unique manner to a position with *eight* rooks, no two of which attack each other (or occupy the same square).

There are 8! positions of the latter kind. Indeed, the rook in the first column may be placed in any of eight different squares; after choosing his position, the rook in the second column may be placed in any of the seven squares that aren't in the same row as the first rook; and so on.

Thus, each position of seven rooks may be obtained by taking one of these 8! positions and removing one of the 8 rooks. This results in a total of $8 \cdot 8! = 322,560$ different positions.

Problem 4. Recall that $\lceil x \rceil$ denotes the least integer greater than or equal to x. For what integer n do we have

$$\left\lceil \sqrt{1! + 2! + 3! + \cdots + n!} \right\rceil = 2010\,?$$

(A) 7 (B) 8 (C) 9 (D)$^\heartsuit$ 10 (E) 11

Solution. Obviously we're going to get this by order of magnitude estimates. Since $n! < 1! + 2! + \cdots + n! < 2 \cdot n!$, we want $\sqrt{n!}$ to be of the order of magnitude of 10^3. Since $\sqrt{5!}$ has the order of magnitude of 10, we start with 9! and note that

$$\sqrt{9!} = \sqrt{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 9 \cdot 8 \cdot \sqrt{70}\,.$$

Thus $586 < \sqrt{9!} < 648$, whereas

$$\sqrt{10!} = \sqrt{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 10 \cdot 9 \cdot 8 \cdot \sqrt{7}\,,$$

and so $1440 < \sqrt{10!} < 2160$. $n = 10$ must be the one that works, as $n = 11$ will put us over the top.

Problem 5. Let

$$x = 1 + \frac{1}{2 + \frac{1}{3}}, \qquad y = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}, \qquad \text{and} \qquad z = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}}.$$

Which of the following is correct?

(A) $x < y < z$ (B)$^\heartsuit$ $x < z < y$ (C) $y < x < z$ (D) $y < z < x$ (E) $z < y < x$

Solution. When we increase the denominator of a fraction, of course the number decreases. So, when we increase the denominator of a fraction that is in a denominator, the resulting number increases, and so on. Thus, $x < z$, since $3 < 3 + \frac{1}{4+\frac{1}{5}}$. Similarly, since $3 + \frac{1}{4+\frac{1}{5}} < 3 + \frac{1}{4}$, $z < y$.

Connections. This problem is motivated by continued fractions and how they are alternately higher and lower than the real number that they are approximating. In fact, y and z differ by only $1/4710 \approx 0.0002$.

Problem 6. We all know that $\sin(0°) = \sin(0)$ (where the latter input is in radians). What is the smallest *positive* number x so that $\sin(x°) = \sin(x)$?

(A) π (B) 180 (C) $\dfrac{360\pi}{180 - \pi}$ (D)$^\heartsuit$ $\dfrac{180\pi}{180 + \pi}$ (E) $\dfrac{360\pi}{180 + \pi}$

Solution. Remember that $\sin(x) = \sin(x + n\pi)$ for any *even* integer n, but is also equal to $\sin(n\pi - x)$ for any *odd* integer n. So, converting degrees to radians, we must solve

$$x \cdot \frac{\pi}{180} = x + n\pi \ (n \text{ even}) \qquad \text{or} \qquad x \cdot \frac{\pi}{180} = n\pi - x \ (n \text{ odd}).$$

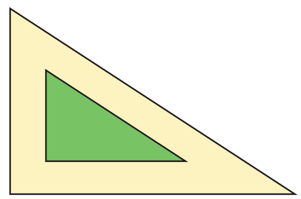
These give, respectively, $x = \dfrac{-180n\pi}{180 - \pi}$ for n even or $x = \dfrac{180n\pi}{180 + \pi}$ for n odd. Clearly the smallest positive solution comes by taking $n = 1$.

Problem 7. A phone number is *cool* if it is either of the form *abc-abcd* or of the form *abc-dabc* (or both) for some digits $a \neq 0$, b, c, and d. If numbers are assigned randomly, what is the chance that you will get a cool phone number? (Note: For the purposes of this problem, the first digit of any phone number cannot be 0, but there is no such restriction on the remaining digits.)

(A) $1/2$ (B) $2010/10^6$ (C)$^\heartsuit$ $1999/10^6$ (D) $1/500$ (E) None of the above

Solution. Denote the fourth, fifth, sixth, and seventh digits of the phone number by x_j, $j = 4, 5, 6, 7$. The chance of getting a phone number of the first form is $(1/10)^3 = 1/1000$, since, whatever a, b, c may be the chance that $x_5 = a$ is $1/10$, the chance that $x_6 = b$ is $1/10$, and the chance that $x_7 = c$ is $1/10$. A similar argument holds for the second form. However, we must ask whether the two forms can coincide. This happens if and only if $a = b = c = x_4 = x_5 = x_6 = x_7$, and there's a $(1/10)^6$ chance that this happens (whatever a may be, the other six must be chosen the same). So the final answer is $P = \dfrac{1}{1000} + \dfrac{1}{1000} - \dfrac{1}{10^6} = \dfrac{1999}{10^6}$.

Problem 8. Derek prefers his brownies from the center of the pan, and Jacob prefers them from around the edge. Their friend Ellie gives them a pan of brownies in the form of a 3-4-5 right triangle. How far from the edges should Derek and Jacob cut it so that they each get equal areas of brownies?



(A)$^\heartsuit$ $1 - 1/\sqrt{2}$ (B) $1/3$ (C) $1/\sqrt{2}$ (D) $2/3$ (E) 1

Solution. The easiest solution is this: Observe that the inscribed circle of a 3-4-5 triangle has radius $\dfrac{\text{area of triangle}}{\text{semiperimeter}} = 1$. If Derek and Jacob cut out a similar triangle a distance x from each edge, its inscribed circle will have radius $1 - x$. Since the ratio of the areas is $\dfrac{(1-x)^2}{1} = \dfrac{1}{2}$, we must have $1 - x = 1/\sqrt{2}$ (of course, $1 - x > 0$).

Comment:. This argument does not work on the brownie problem in the ciphering, #9, because there is no inscribed circle for a non-square rectangle.

Problem 9. If q, r, and s are the solutions of $x^3 - 5x^2 + 7x + 4 = 0$, then what is $q(r^2 + s^2) + r(s^2 + q^2) + s(q^2 + r^2)$?

(A) -47 (B) -23 (C) 23 (D)$^\heartsuit$ 47 (E) not enough information

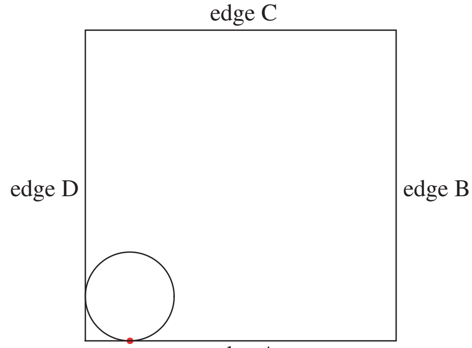
Solution. If $x^3 - 5x^2 + 7x + 4 = (x - q)(x - r)(x - s)$, then we have

$$\begin{aligned} q + r + s &= 5 \\ qr + qs + rs &= 7 \\ qrs &= -4\,. \end{aligned}$$

Therefore, $q(r^2 + s^2) + r(s^2 + q^2) + s(q^2 + r^2) = (q + r + s)(qr + rs + sq) - 3qrs = 5(7) - 3(-4) = 47$.

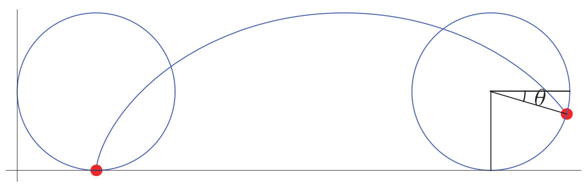
(In general, any symmetric polynomial in the solutions of a polynomial can be expressed in terms of the coefficients of the polynomial.)

Problem 10. A circle of radius 1 is sitting inside a 7×7 square, tangent to the left and bottom edges, as pictured, with point P at the point of contact with edge A. If the circle rolls around the inside of the square without slipping, then which edge does the point P next touch?



(A) A (B)$^\heartsuit$ B (C) C (D) D (E) It never touches an edge again.

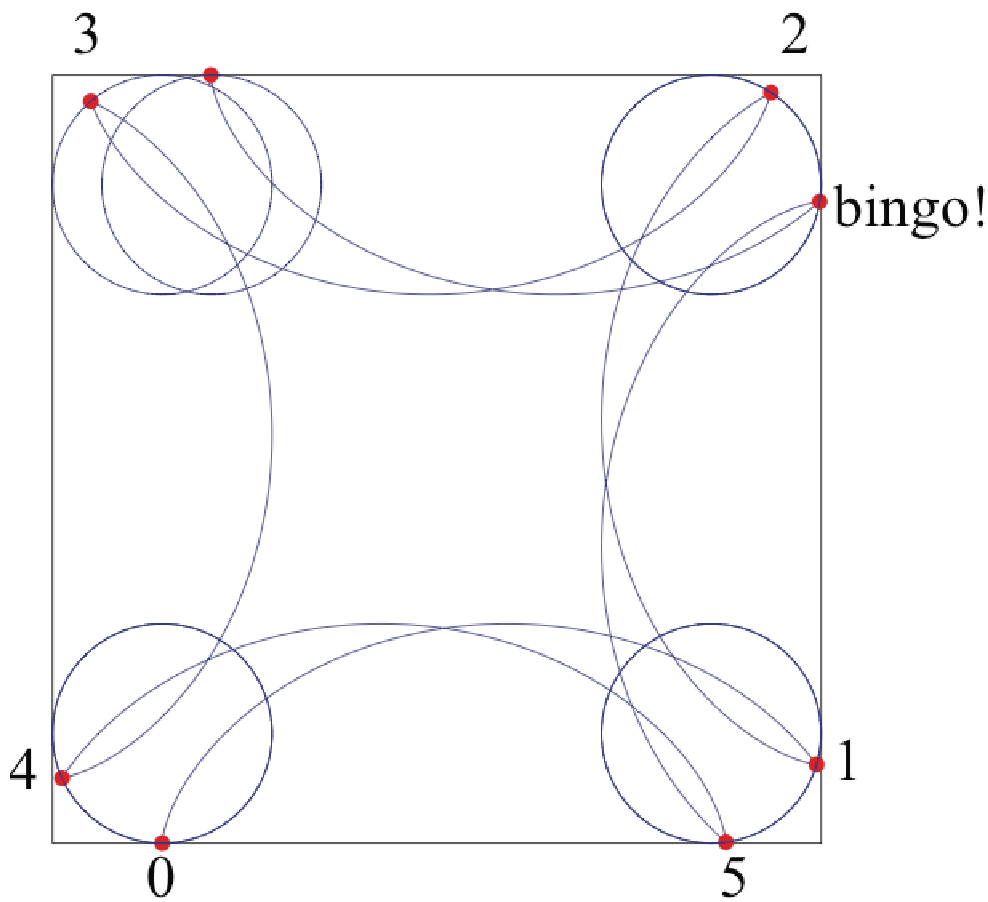
Solution. The point P traces out arcs of cycloids. It is not too difficult to write out parametric equations (if, for example, one wishes to get Mathematica images such as the ones below).



However, we can solve the problem without any explicit equations. Note that the circle rolls a distance of 5 units across edge A, and so it turns through an angle of 5 radians. This is slightly greater than $3\pi/2$; indeed, $\theta = 5 - 3\pi/2 \approx 0.29 < \pi/10$. Note now that when the circle rolls along edge B, the point P starts to the left of the edge and below the horizontal by θ. Since $5 + \theta$ is still less than 2π, the point P still does not make contact with the edge as it rolls.

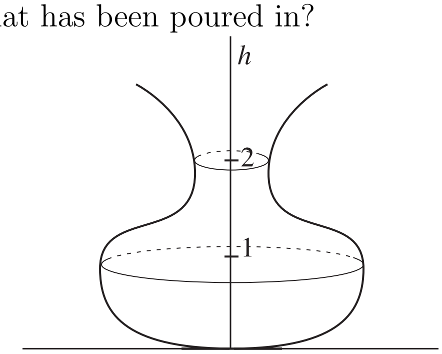
Now, $5\theta < \pi/2$ but $6\theta > \pi/2$. This means that when the circle arrives at the *sixth* (upper right) corner, the point P makes a positive angle (measured clockwise) with the horizontal; therefore, just before this, P must have touched the edge. Thus, P next touches edge B, on the second trip around. See the picture below (note that the picture is too small to see that at position 5 the point is actually not touching).

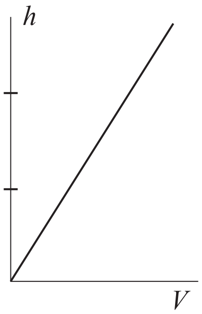
Note: The Mathematica notebook that gives this animation is available at `http://www.math.uga.edu/~shifrin/RollingBall.nbp`. Copy and save this (what will appear as a text file) on your own computer as a file with the same name. To view it, you will need either Mathematica or Mathematica Player (the latter available for free at `http://www.wolfram.com/products/player/download.cgi`).

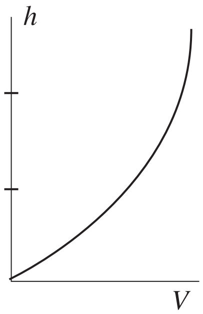


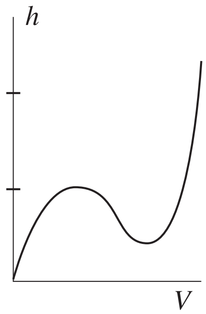
2 Medium Problems

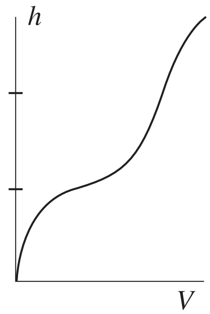
Problem 11. A vase in the shape shown below is slowly filled with water. Which of the graphs below most closely represents the height h of the water as a function of the volume V of water that has been poured in?

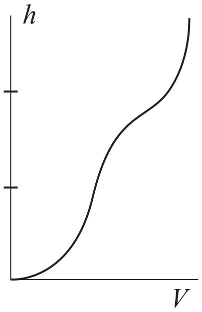


(A) 

(B) 

(C) 

(D)$^\heartsuit$ 

(E) 

Solution. Let's think about the effect that adding a small volume ΔV of water has on the height. First of all, we eliminate (C), since h must always increase as V increases. To begin with, for $0 \le h \le 1$, the radius r is an increasing function of h, so the height rises more quickly for small h than for larger h; i.e., the rate of change of h is decreasing. This is exactly what it means for the graph of h to be concave down. This eliminates (A), (B), and (E). That leaves only (D). (Notice also that $h \approx r^2$ for small r. Since $\Delta V/\Delta h \approx \pi r^2$, we have $\Delta V/\Delta h \approx \pi h$, and so $\Delta h/\Delta V \approx 1/(\pi h)$ is very big near $h = 0$. That confirms the choice of (D).)

Of course, some calculus wouldn't hurt on this problem. Recall that $dV/dh = \pi r^2$, where r is the cross-sectional radius, and so $dh/dV = 1/(\pi r^2) > 0$. This means that h is always increases as a function of V, and so (C) is ruled out. It's evident that the graph of h as a function of V should have infinite slope at the origin (since $dV/dh = 0$ at $h = 0$), and this rules out (A), (B), and (E). This leaves only (D). (We leave it as an exercise to relate the concavity of $h(V)$ to dr/dh.)

Problem 12. Let a_n be the n-th smallest positive integer the sum of whose decimal digits is 3. For example, $a_{18} = 2010$. How many digits does a_{1000} have?

(A) 15 (B) 16 (C) 17 (D)$^\heartsuit$ 18 (E) 19

Solution. The number of such integers with d digits or fewer is $\binom{d+2}{3}$. Indeed, consider placing $(d-1)$ "digit separators" and 3 "digits" in all of the possible orders. Then one can build up a correspondence between the $\binom{(d-1)+3}{3}$ different placement orders and numbers whose decimal digits sum to 3. For example, consider $d = 3$:

$$\begin{array}{ll} ||ooo \mapsto 003 & o|o|o \mapsto 111 \\ |o|oo \mapsto 012 & o|oo| \mapsto 120 \\ |oo|o \mapsto 021 & oo||o \mapsto 201 \\ |ooo| \mapsto 030 & oo|o| \mapsto 210 \\ o||oo \mapsto 102 & ooo|| \mapsto 300 \end{array}$$

Thus we need only find the smallest value of d such that $\binom{d+2}{3} = \frac{d(d+1)(d+2)}{6} \geq 1000$. We find that $\binom{17+2}{3} = 969 < 1000$ but $\binom{18+2}{3} = 1140 \geq 1000$, so the answer is 18.

Problem 13. Let $\sigma(n)$ be the sum of the positive divisors of n. For example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. How many integers n are there so that $\sigma(n) = 72$?

(A) 1 (B) 2 (C) 3 (D) 4 (E)$^{\heartsuit}$ 5

Solution. Let the notation $p^k \mid\mid n$ mean that p^k is the largest power of the prime p that divides n. Then we may express

$$\sigma(n) = \prod_{p^k||n} \left(1 + p + \cdots + p^k\right) = \prod_{p^k||n} \frac{p^{k+1} - 1}{p - 1}.$$

So we are looking for expressions of the form $1 + p + \cdots + p^k$ that divide $72 = 2^3 \cdot 3^2$. It turns out that all such expressions are of the form $1+p$, and they are the following:

$$\begin{array}{ll} 1 + 2 = 3 & 1 + 11 = 12 \\ 1 + 3 = 4 & 1 + 17 = 18 \\ 1 + 5 = 6 & 1 + 23 = 24 \\ 1 + 7 = 8 & 1 + 71 = 72 \end{array}$$

In order to get $\sigma(n) = 72$, they may be combined into the following five numbers:

$$30 = 2 \cdot 3 \cdot 5 \qquad 46 = 2 \cdot 23 \qquad 51 = 3 \cdot 17 \qquad 55 = 5 \cdot 11 \qquad 71 = 71$$

Problem 14. How many different Hamiltonian cycles are there on the vertices of the cube? That is, how many different ways are there to order the vertices of the cube as a cycle $(v_1, \ldots, v_8)$ so that consecutive vertices are adjacent (including v_8 and v_1)? Note that any cycles having the same adjacent vertices are considered the same: e.g., $(v_1, v_2, \ldots, v_8)$, $(v_2, v_3, \ldots, v_8, v_1)$, $(v_3, \ldots, v_8, v_1, v_2)$, etc., as well as $(v_8, \ldots, v_1)$, $(v_7, \ldots, v_1, v_8)$, etc.

(A) 1 (B) 2 (C) 3 (D) 4 (E)$^{\heartsuit}$ 6

Solution. One can either simply count the cycles, or use some sort of ad-hoc argument like the following:

You can identify two special faces associated to any Hamiltonian cycle, for example, by looking at the two faces that only have two edges in the cycle. These two special faces are opposite, and there are two ways in which two opposite faces can correspond to a Hamiltonian cycle. Thus, there are two cycles for each pair of opposite faces, for a total of six.

Problem 15. Let S be the sum of all seven-digit numbers whose digits are some permutation of 1, 2, 3, 4, 5, 6, and 7. Find the next-to-last ("tens") digit of S.

(A) 0 or 1 (B) 2 or 3 (C) 4 or 5 (D)$^\heartsuit$ 6 or 7 (E) 8 or 9

Solution. There are 7! such numbers. They may be paired so that each pair sums to 8888888, so their sum S is $7! \cdot 4444444$.

This number is divisible by 10, but after dividing through by 10, we obtain $S/10 = 7 \cdot 6 \cdot 4 \cdot 3 \cdot 4444444$. We need only figure out the ones digit of this product, which ends up being 6.

The sum is actually 22399997760.

Problem 16. Suppose there are 10 points on the circumference of a circle. Draw all $\binom{10}{2} = 45$ chords connecting these points. What is the largest number of regions into which these chords can divide the circle?

(A) 128 (B)$^\heartsuit$ 256 (C) 512 (D) 1024 (E) None of the above

Solution. Suppose there are n points instead of 10. Then if the points on the circle are in as general a position as possible (no three chords coincide), there are $\binom{n}{4}$ intersection points inside the circle. Thus the number of vertices in the diagram is $n + \binom{n}{4}$. The number of edges is half the sum of the degrees of the vertices, which is $n(n-1) + 4\binom{n}{4}$. Finally, by Euler's formula, we can compute the number of regions in the circle:

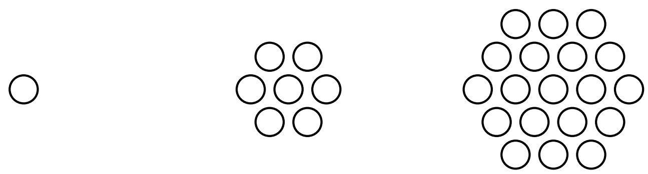
$$\begin{aligned}
1 + E - V &= 1 + \frac{n(n-1)}{2} + 2\binom{n}{4} - \binom{n}{4} \\
&= \binom{n}{0} + \binom{n}{2} + \binom{n}{4} \\
&= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}.
\end{aligned}$$

In the case $n = 10$, this is half of the binomial coefficients in the 9th row of Pascal's triangle, so the answer is $\frac{1}{2}2^9 = 2^8 = 256$.

Alternate solution. There is also a combinatorial proof of this result which assigns subsets of $\{1, \ldots, n-1\}$ of size at most four to each region of the subdivision by chords. See, for example, the section "How many regions?" in *The Book of Numbers* by Conway and Guy.

Remark. Note that for the first few values of n, the number of regions is 1, 2, 4, 8, and 16—all powers of 2. However, the next value is 31. The value 256 is again a power of 2, but it is 1 smaller than what you would expect if the original pattern continued.

Problem 17. The first three centered hexagonal numbers ($h_1 = 1$, $h_2 = 7$, and $h_3 = 19$) are illustrated below:

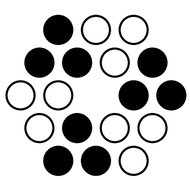


That is, the n-th centered hexagonal number h_n is the number of circles in a diagram that has one circle surrounded by $(n-1)$ layers of circles in a hexagonal lattice. What is the 10-th centered hexagonal number h_{10}?

(A) 217 (B) 231 (C) 276 (D) 331 (E)$^\heartsuit$ None of the above

Solution. The correct answer is 271. In general, the n-th centered hexagonal number is $n^3 - (n-1)^3 = 3n^2 - 3n + 1 = 1 + 6 \cdot \frac{n(n-1)}{2}$.

This final formula can be seen visually as one more than six times a triangular number:



Alternate solution. You can also see the expression $n^3 - (n-1)^3$ by interpreting each centered hexagonal number as the extra layer you have to add to an $(n-1)$-cube to obtain an n-cube.

Problem 18. What is the smallest positive integer that can **not** be written as the sum of 10 or fewer factorials?

(A) 119 (B) 163 (C)$^\heartsuit$ 239 (D) 241 (E) $10! - 1$

Solution. We can actually think of representing a positive integer by a unique numeral in "base factorial." Since $(n+1)n! = (n+1)!$, we will use n or fewer n!s. With this restriction, every positive integer can be written *uniquely* as a sum of factorials, with k or fewer k!s for each k. For example,

$$87 = 3 \cdot 4! + 2 \cdot 3! + 1 \cdot 2! + 1 \cdot 1! = \text{“}3211_!\text{”} .$$

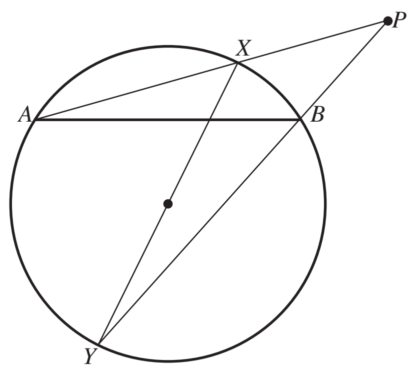
What guarantees that this works is the formula

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 ,$$

often assigned as an elementary exercise in mathematical induction.

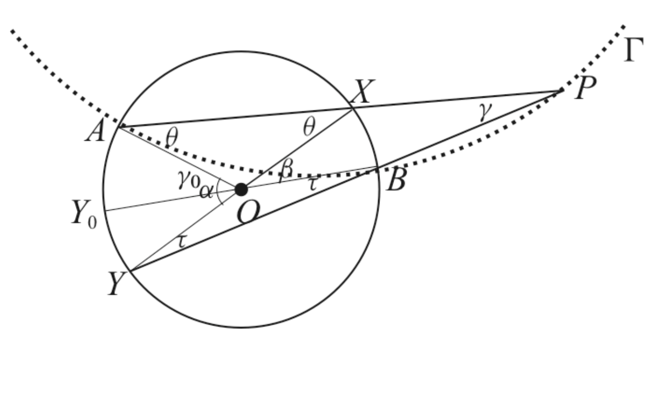
We are looking for the smallest number so that its digits in base factorial add up to 11. Clearly this is the number $14321_! = 5! + 4 \cdot 4! + 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1! = 239$.

Problem 19. Points A and B are fixed on a circle, and $\overline{AB}$ is not a diameter. Consider a diameter $\overline{XY}$ and the point P given by the intersection of $\overleftrightarrow{AX}$ and $\overleftrightarrow{BY}$, as pictured. What is the locus of all such points P as X moves all the way around the circle? (Note: When $A = X$, we have $P = A$, and when $B = Y$, we have $P = B$.)



(A) an arc of a circle (B) two arcs of a circle, not forming a complete circle
(C) an ellipse that is not a circle (D)$^\heartsuit$ a circle (E) None of the above

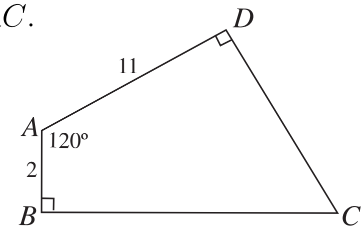
Solution. Let Y_0 denote the point opposite B on the circle. When Y is outside $\widehat{AY_0}$, we note that P is outside the circle, and when Y lies in $\widehat{AY_0}$, P moves inside the circle. Let's start with P outside. We begin by recalling what used to be a well-known result in high school geometry: As shown in the figure, we have



$\gamma = (\alpha - \beta)/2$. (Proof: Considering the angle sum of quadrilateral $OBPX$ and the sum of the angles at O we get $\beta + \gamma = \theta + \tau = (\alpha + \beta)/2$.) In our application, since $\overline{XY}$ is always a diameter, we have $\alpha - \beta = \gamma_0$, independent of the position of P. Therefore, $\gamma = \gamma_0/2$ is a constant as P varies. Since $\angle APB$ is a constant, P traces out the arc of a circle Γ with chord $\overline{AB}$. (Why?)

But, now, what happens when P comes inside the circle? An analogous argument to the one we gave above shows that $\pi - \gamma = (\alpha + \beta)/2$, and so now $\pi - \gamma = \gamma_0/2$ (draw the picture and check!). This means that $\angle PAB$ now subtends the opposite arc on the circle Γ. Thus, the desired locus is the entire circle Γ.

Problem 20. Given quadrilateral $ABCD$, as pictured, with $\angle A = 120°$, and both $\angle B$ and $\angle D$ right angles. If $AB = 2$ and $AD = 11$, find AC.



(A) $5\sqrt{3}$ (B) $8\sqrt{3}$ (C)$^\heartsuit$ 14 (D) $10\sqrt{2}$ (E) None of the above

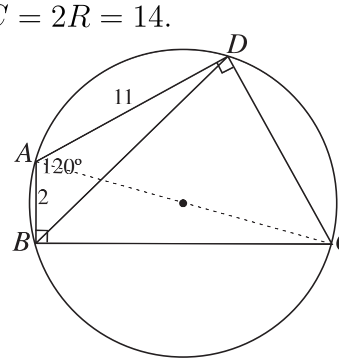
Solution. The first observation is that this quadrilateral is (co)cyclic, i.e., its vertices lie on a circle. Since $\angle B = 90°$, $\overline{AC}$ will in fact be a diameter of the circle, as shown below. To find the radius of the circle, we use the somewhat esoteric formula for the circumradius of a circle with sidelengths a, b, and c and area A: $R = \dfrac{abc}{4A}$. First, we use the law of cosines to find that

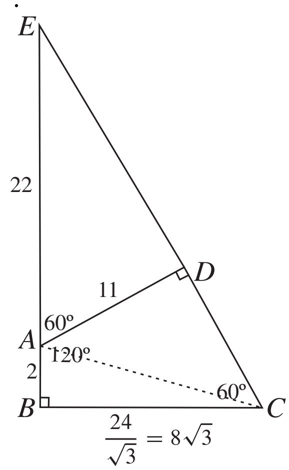
$$BD = \sqrt{2^2 + 11^2 - 2 \cdot 2 \cdot 11 \cdot \cos(120°)} = \sqrt{125 + 22} = \sqrt{147} = 7\sqrt{3}\,.$$

Now, we find the area of $\triangle ABD$ either by using Heron's formula or, more simply, by using the original information: $A = \frac{1}{2}(2)(11)(\sin 120°) = 11\sqrt{3}/2$. Therefore, the circumradius of $\triangle ABD$ is

$$R = \frac{2 \cdot 11 \cdot 7\sqrt{3}}{2 \cdot 11\sqrt{3}} = 7\,.$$

Finally, $AC = 2R = 14$.





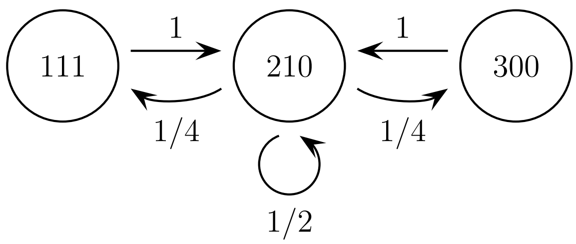
Second (and more elementary) solution..

Cleverly extend $\overline{BA}$ and $\overline{CD}$ to intersect at point E. Then basic 30°-60°-90° triangle facts give $AE = 22$ and hence $BC = 24/\sqrt{3} = 8\sqrt{3}$. Then Pythagoras gives $AC = \sqrt{2^2 + (8\sqrt{3})^2} = 2\sqrt{1 + 16 \cdot 3} = 14$.

Problem 21. One marble is placed in each of three bowls. Five times in succession, a marble is moved from one bowl (chosen at random) to a different bowl (chosen at random). What is the probability that we again have one marble in each of the three bowls?

(A) 5/108 (B) 1/18 (C) 1/6 (D)$^\heartsuit$ 5/32 (E) 1/9

Solution. The three possible positions can be called 111, 210, and 300, corresponding to zero, one, and two empty bowls. The transition graph is as follows:



In other words, if you're either in 111 or 300, then the next state will necessarily be 210. However, if you're in state 210, then you either stay or leave with equal probability; if you leave, then it's equal probability either way. (In particular, the overall probability of going from 210 to 111 is 1/4.)

Thus the first step is fixed, always to 210, and the last step will have to be from 210 to 111 if we intend to return to the initial configuration. Hence we have to analyze the probability that if you start at 210 and take three steps you end up back at 210. It turns out to be 5/8 here by simply calculating the different possibilities. (One can either stay at 210 for three steps at a probability of 1/8, or one can stay for one of the turns but leave for the other two for a probability of 1/4 in two different ways. Note that for this calculation, it's possible to consider states 111 and 300 the same because they act the same; thus, there are only two states, "at 210" and "away from 210 for one turn".) After multiplying by the 1/4 probability for the last move, the final answer is 5/32.

With some standard recursion analysis, one can find that the probability of returning after n steps instead of 5 is

$$\frac{1}{3}\left(\frac{1}{2} + \left(-\frac{1}{2}\right)^n\right),$$

so it generally hovers around 1/6. In fact, it is the fraction with denominator 2^n that is nearest to 1/6.

3 Hard Problems

Problem 22. Now, *three* marbles are placed in each of three bowls. Five times in succession, a marble is moved from one bowl (chosen at random) to a different bowl (chosen at random). What is the probability that we again have three marbles in each of the three bowls?

(A)$^{\heartsuit}$ 5/108 (B) 1/18 (C) 1/6 (D) 5/32 (E) 1/9

Solution. Notice that there's no difference between starting with three marbles in each bowl and starting with six or ten marbles. We can never move more than two marbles out of any bowl if we are to have a chance of returning to our original configuration in five moves.

Because there are 5 moves, there will be precisely *one* bowl from which only *one* marble is removed. Once we know which bowl that is and where that marble is put, all the other moves are determined (up to order). Denote by $[ij]$ the event of moving one marble from bowl i to bowl j, $i \neq j$. Suppose, for example, that [12] is the only move which removes a marble from bowl 1. The two marbles removed from bowl 3 cannot both be put in bowl 2 because we have already put a marble from bowl 1 there; nor can both marbles from bowl 3 be put in bowl 1 because only one marble is removed from that bowl. Thus, both moves [31] and [32] must appear. At this point, there are 5 marbles in bowl 2 and 1 marble in bowl 3. To get back to the original configuration, both remaining moves must be [23]. (From a slightly different perspective, up to reordering, the 5 moves must consist of a "3-cycle" [12][23][31] followed by a "2-cycle" [23][32].)

Now let's count. We must choose the bowl from which only one marble is removed; there are 3 choices for this. We must choose to which bowl we're transferring that marble; there are 2 choices for that. Now, up to order, the remaining moves are determined. There are $5!/2! = 60$ permutations of this sequence of moves (because precisely one of the moves appears twice). Since each move has probability $1/6$ (here it is crucial that we never have an empty bowl), the answer is $(6 \cdot 60)/6^5 = 5/108$.

Alternate solution (in the spirit of no.21).

Using the notation of the solution of #21, we start with the configuration 333 and, with probability 1, go to the configuration 234 with the first move. Obviously, with the fifth move, we must return from 234 to 333, and there's a probability of $1/6$ that we do so. So it remains to find the probability of starting at 234 and returning to that configuration in a sequence of three moves. From 234 we can arrive at 144, 135, or 225, and we must stay among these 5 configurations if there's any chance of finishing in the desired number of moves. Now we write down the transition matrix (whose ij-entry gives the probability of moving from configuration j to configuration i): Ordering the configurations by 333, 234, 144, 135, and 225, we have

$$A = \begin{bmatrix} 0 & 1/6 & 0 & 0 & 0 \\ 1 & 1/3 & 1/3 & 1/6 & 1/3 \\ 0 & 1/6 & 0 & 1/6 & 0 \\ 0 & 1/6 & 1/3 & 0 & 1/3 \\ 0 & 1/6 & 0 & 1/6 & 0 \end{bmatrix}$$

Now the 22-entry of A^3 (which is actually not that hard to compute if we factor $1/6$ out of A) gives us the probability of returning to 234 in three moves, and this entry is $\ldots 5/18$. Thus, the answer, as promised, is $\frac{1}{6} \cdot \frac{5}{18} = \frac{5}{108}$. By the way, we suggest that the studious reader recast the argument in the solution of #21 using a transition matrix.

Problem 23. Suppose triangle ABC has side lengths $BC = 13$, $AC = 14$, and $AB = 15$. Extend the two sides meeting at vertex A by BC, the two sides meeting at vertex B by AC, and the two sides meeting at vertex C by AB. The endpoints of these six new line segments all lie on a circle. The radius of this circle is $\sqrt{n}$ for some integer n. What is the sum of the digits of n?

(A)$^\heartsuit$ 16 (B) 17 (C) 18 (D) 19 (E) 20

Solution. This circle is called the *Conway circle*. Its center is the same as the incenter of $\triangle ABC$ and its radius is $\sqrt{r^2 + s^2}$, where r and s are the inradius and semiperimeter of ABC, respectively. In this case, we may compute that $s = \frac{13+14+15}{2} = 21$. Also, the inradius $r = A/s$, where A is the area of the triangle. We may either use Heron's formula to compute the area, or notice that a 13-14-15 triangle is composed of a 5-12-13 right triangle stuck to a 9-12-15 right triangle and so $A = 84$. This $r = 4$.

Finally, we get that

$$\sqrt{4^2 + 21^2} = \sqrt{457},$$

so the sum of the digits of n is $4 + 5 + 7 = 16$.

We can prove the properties of Conway's circle mentioned above fairly easily. Let I be the incenter of the triangle, and let X be the point at the end of the extension (at vertex A) of $\overline{AB}$ by length BC. Finally, suppose a perpendicular from point I to side $\overline{AB}$ hits the side at P. Then we have

$$IX = \sqrt{IP^2 + PX^2} = \sqrt{IP^2 + (PA + AX)^2} = \sqrt{r^2 + (s - BC + BC)^2} = \sqrt{r^2 + s^2}.$$

But this argument works symmetrically for any of the six points, so the circle of the given radius centered at the incenter passes through all six points.

Problem 24. In how many ways can 2010 be written as a sum of 2 or more *consecutive* positive integers? (For example, $9 = 4 + 5 = 2 + 3 + 4$ can be so written in 2 ways.)

(A) 1 (B) 3 (C) 5 (D)$^\heartsuit$ 7 (E) 8

Solution. First notice that for any k, any positive integer N can be written in at most one way as the sum of k consecutive positive integers, and the largest possible such k would occur when $N = 1+2+\cdots+k = k(k+1)/2$. So, if we try to solve this for $N = 2010$, we find $k(k+1) = 4020$, and so $k < \sqrt{4020} < 64$.

Next notice that if N is the sum of *any* k consecutive integers, then

$$N \equiv 1+\cdots+k \equiv \frac{k(k+1)}{2} \pmod{k}, \quad \text{so} \quad N \equiv \begin{cases} 0 \pmod{k}, & k \text{ odd} \\ k/2 \pmod{k}, & k \text{ even} \end{cases}. \quad (*)$$

Conversely, if $(*)$ holds, then let $m = \dfrac{N-(1+2+\cdots+k)}{k}$, and notice that $N = (m+1)+(m+2)+\cdots+(m+k)$. Thus, we need only find the number of k so that $(*)$ holds.

For k odd, k must be a factor of $N = 2010 = 2\cdot 3\cdot 5\cdot 67$. This gives 3, 5, and 15 as odd factors of 2010 that are less than 64.

Now we want to determine for which even numbers k we have $2010 \equiv k/2 \pmod{k}$. Note that this can occur only if 2010 is divisible by $k/2$ and that $k/2$ must be even. This gives us only $k = 4$, $k = 12$, $k = 20$, and $k = 60$. (Since 2010 is not divisible by 4, these conditions are in fact equivalent.)

Finally, let's check that we do in fact get solutions in all these 7 cases:

k	m	$2010 =$
3	668	$669+670+671$
4	500	$501+502+503+504$
5	399	$400+401+402+403+404$
12	161	$162+\cdots+173$
15	126	$127+\cdots+141$
20	90	$91+\cdots+110$
60	3	$4+\cdots+63$

Problem 25. Let $V_n(r)$ be the (n-dimensional) volume of the n-dimensional ball of radius r. So, $V_2(r) = \pi r^2$, $V_3(r) = \frac{4}{3}\pi r^3$, $V_4(r) = \frac{\pi^2}{2}r^4$, etc. We also set $V_0(r) = 1$. Find the sum

$$S = V_0(1) + V_2(1) + V_4(1) + \ldots$$

(A)$^\heartsuit$ e^π (B) π^e (C) π^2 (D) infinite (the sum diverges) (E) None of the above

Solution. The volume of an n-dimensional ball of radius r is given by the following very simple but not widely known formula, which can be proved by induction using some calculus:

$$V_n(r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})!} = \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!}r^n$$

(Note that $V_n(r) = V_n(1) \cdot r^n$ by dimension analysis.) So,

$$V_0(r) + V_2(r) + V_4(r) + \cdots = 1 + \frac{(\pi r^2)^1}{1!} + \frac{(\pi r^2)^2}{2!} + \frac{(\pi r^2)^3}{3!} + \cdots = e^{\pi r^2}$$

using the formula

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

For $r = 1$, this gives e^π.

Note: This formula works for odd n as well, if you take into account that $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$. For example, $(\frac{3}{2})! = \frac{3}{2} \cdot (\frac{1}{2})! = \frac{3}{4}\sqrt{\pi}$, and so

$$V_3(r) = \frac{\pi^{\frac{3}{2}}}{(\frac{3}{2})!} r^3 = \frac{\pi^{\frac{3}{2}}}{\frac{3}{4}\sqrt{\pi}} r^3 = \frac{4\pi}{3} r^3.$$

For completeness, we include a proof of the formula for the $(2n)$-volume of a $(2n)$-ball, by induction (using calculus). The base case $V_0(r) = 1$ is by convention. For the inductive step, we have in polar coordinates

$$\begin{aligned} V_{2n+2}(1) &= \int_0^1 \int_0^{2\pi} V_{2n}\left(\sqrt{1-r^2}\right) r\, d\theta\, dr \\ &= 2\pi \int_0^1 V_{2n}\left(\sqrt{1-r^2}\right) r\, dr \\ &= 2\pi \int_0^1 \frac{(\pi(1-r^2))^n}{n!} r\, dr \\ &= \frac{2\pi^{n+1}}{n!} \cdot \frac{-(1-r^2)^{n+1}}{2(n+1)}\Bigg|_{r=0}^{r=1} \\ &= \frac{\pi^{n+1}}{(n+1)!} \end{aligned}$$

Connections. The celebrated Gelfond-Schneider Theorem implies that e^π is transcendental. So e^π is known to be transcendental, whereas none of $e + \pi$, $e - \pi$, $e \cdot \pi$, e/π, and π^e is even proven to be irrational.

Authors. Jacob Rooney contributed 4, Mo Hendon contributed 5, 6, 8, 18, 24, Jacob Rooney and Derek Ponticelli contributed 22.

Solutions for 4, 5, 6, 8, 18, 19, 20, 24 were written by Ted Shifrin.

Problems and solutions for 1, 3, 12, 13, 14, 15, 16, 17, 21, 23, 25 are by Boris Alexeev, with contributions by Valery Alexeev.

Sources. 19 was taken from an old USA Olympiad, and 20 is inspired by an old AHSME problem.