Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 21, 2017

WITH SOLUTIONS

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. Suppose $a_n > 0$ for every integer n, and that $a_{n+1} = a_n^2 - 2^n$. If $a_4 = 2017$, what is a_0?

(A) 5 (B) 4 (C) 3 (D)$^\heartsuit$ 2 (E) 1

Solution. Work backwards:

$$\begin{aligned} 2017 = a_3^2 - 2^3 &\implies a_3^2 = 2025 \implies a_3 = 45 \\ 45 = a_2^2 - 2^2 &\implies a_2^2 = 49 \implies a_2 = 7 \\ 7 = a_1^2 - 2^1 &\implies a_1^2 = 9 \implies a_1 = 3 \\ 3 = a_0^2 - 2^0 &\implies a_0^2 = 4 \implies a_0 = 2. \end{aligned}$$

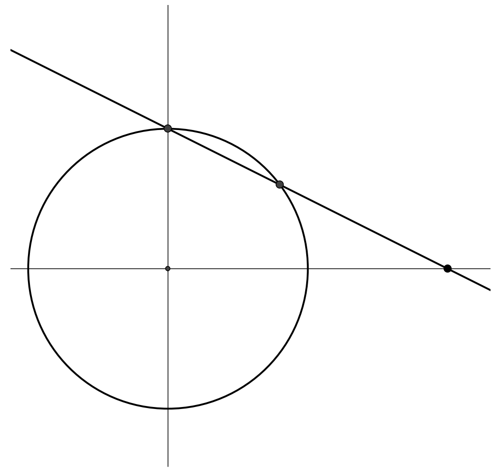
Problem 2.

Of A and B this is the lore:
When added they make 24.
If A over 3
is A over B,
what's $A + 2B$ plus 2 more?

(A) 28 (B) 29 (C)$^\heartsuit$ 29 or 50 (D) 47 (E) 50

Solution. We are given the equations $A + B = 24$ and $\frac{A}{B} = \frac{A}{3}$. Rearranging the second equation gives $A(B-3) = 0$, so $B = 3$ or $A = 0$. In the first case $A+3 = 24$, so $A = 21$ and $A + 2B + 2 = 29$. In the second case $B + 0 = 24$, so $B = 24$ and $A + 2B + 2 = 50$.

Problem 3. Draw the circle $x^2+y^2 = 1$, then draw the line through the "north pole" $(0, 1)$ meeting the x-axis at $(2, 0)$. What is the x-coordinate of the other point where the line meets the circle?



(A)$^\heartsuit$ $\frac{4}{5}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{2}{3}$ (E) 1

Solution. The equation of the line is $y = -\frac{1}{2}x + 1$. Substituting into $x^2 + y^2 = 1$, we get $x^2 + (-\frac{1}{2}x + 1)^2 = 1$, or $\frac{5}{4}x^2 - x = 0$, which has roots $x = 0$ and $x = \frac{4}{5}$. Notice that the corresponding y-coordinate is

$$y = -\frac{1}{2}\left(\frac{4}{5}\right) + 1 = \frac{3}{5},$$

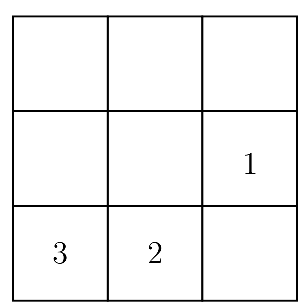
so that the point of intersection is $(\frac{4}{5}, \frac{3}{5})$. This is indeed on $x^2 + y^2 = 1$, because of the identity $3^2 + 4^2 = 5^2$.

Now try this problem again, replacing the point $(2, 0)$ on the x-axis with $(\frac{p}{q}, 0)$ for any rational numbeer $\frac{p}{q}$. You'll find the intersection point to be

$$\left(\frac{2pq}{p^2 + q^2}, \frac{p^2 - q^2}{p^2 + q^2}\right).$$

Up to scaling and reordering of the legs, this parametrizes all Pythagorean triples: $(2pq, p^2 - q^2, p^2 + q^2)$.

Problem 4. In this magic square, there is exactly one way to fill the empty squares so that every row, every column, and both main diagonals add up to the same value. What is that value?



(A) 15/2 (B) 9 (C)$^\heartsuit$ 21/2 (D) 12 (E) 15

Solution. First fill the empty square with variables a, b, c, d, e, f, as shown:

a	b	c
d	e	1
3	2	f

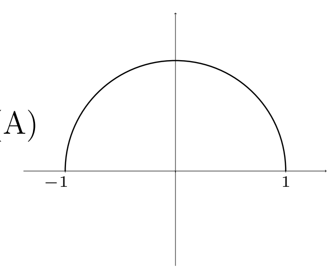
Notice that

$$c + 1 + f = 3 + 2 + f \Rightarrow c = 4,$$
$$3 + e + 4 = 4 + 1 + f \Rightarrow f = e + 2,$$
$$2 + e + b = 4 + 1 + f = 5 + e + 2 \Rightarrow b = 5,$$
$$1 + e + d = 4 + 1 + f = 5 + e + 2 \Rightarrow d = 6,$$
$$3 + d + a = 4 + 1 + f = 5 + e + 2 \Rightarrow 9 + a = 7 + e \Rightarrow a = e - 2,$$
$$a + e + f = c + 1 + f \Rightarrow 3e = e + 7 \Rightarrow e = \frac{7}{2} \Rightarrow f = \frac{11}{2}, a = \frac{3}{2}.$$

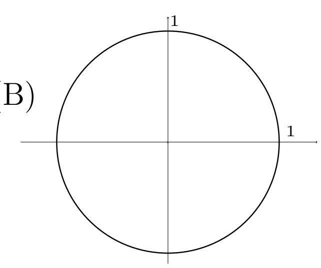
So the common sum is $\frac{3}{2} + \frac{7}{2} + \frac{11}{2} = \frac{21}{2}$.

Problem 5. Which of the following is the graph of $\cos(y) - \sin(x) = 0$?

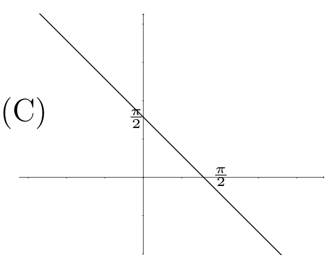
(A)



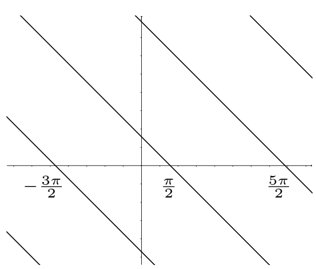
(B)



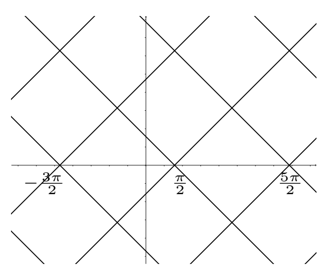
(C)



(D)



(E)$^\heartsuit$



Solution. Notice that

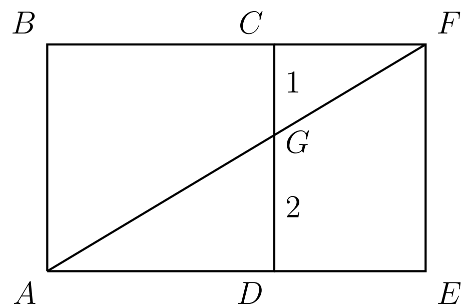
$$\cos(y) - \sin(x) = \sin(\frac{\pi}{2} - y) - \sin(x),$$

and so the solutions include all pairs (x, y) on the lines

$$x + y = \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}.$$

Since cos is even, the set of solutions is also symmetric about the x-axis; this eliminates all possibilities except (E).

Problem 6. Consider the rectangle $ABFE$ as shown in the figure below. $ABCD$ is a square. If $CG = 1$ and $GD = 2$, then what is the perimeter of the rectangle $ABFE$?



(A) $6 + 2\sqrt{5}$ (B) 10 (C) 12 (D)$^\heartsuit$ 15 (E) 24

Solution. Since $ABCD$ is a square and $CD = 3$, we have $AD = 3$. Now, since the triangles CGF and AGD are similar, we get $CF = \frac{3}{2}$. Therefore, the perimeter of the rectangle $ABFE = 2[3 + 3 + \frac{3}{2}] = 15$.

Remark. Keeping $CG = 1$ fixed and the length GD varying, say $GD = x$, one can ask: what is the minimum perimeter of the rectangle $ABFE$ that can be achieved? Using the same method as above shows that $CF = \frac{1+x}{x} = 1 + \frac{1}{x}$ and hence, the perimeter of the rectangle $ABFE = 2[3 + 2x + \frac{1}{x}]$. By the AM-GM inequality,

$$2x + \frac{1}{x} \geq 2\sqrt{2}$$

with equality achieved when $2x = \frac{1}{x}$, that is when $x = \frac{1}{\sqrt{2}}$. Therefore the minimum perimeter of $ABFE = 6 + 4\sqrt{2}$.

Problem 7. Let $f(x)$ and $g(x)$ be real-valued functions defined for all real numbers x. Suppose that for certain constants a, b, A, B, we have that

$$0 < a \leq f(x) \leq A, \quad \text{and} \quad 0 < b \leq g(x) \leq B$$

for all real numbers x. Which of the following must also be true for all x?

(A) $\frac{a}{b} \leq \frac{f(x)}{g(x)} \leq \frac{A}{B}$, (B)$^\heartsuit$ $\frac{a}{B} \leq \frac{f(x)}{g(x)} \leq \frac{A}{b}$, (C) $\frac{A}{b} \leq \frac{f(x)}{g(x)} \leq \frac{a}{B}$,
(D) $\frac{A}{B} \leq \frac{f(x)}{g(x)} \leq \frac{a}{b}$, (E) none of these must be true

Solution. Since $g(x) \geq b$ and $b > 0$, we have that $\frac{1}{g(x)} \leq \frac{1}{b}$. Since $f(x)$ is positive and $f(x) \leq A$, we deduce that

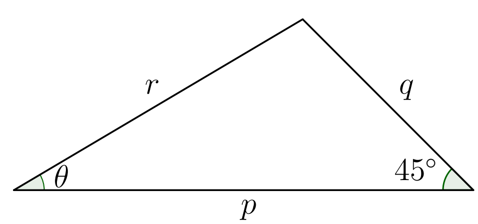
$$\frac{f(x)}{g(x)} \leq f(x) \cdot \frac{1}{b} \leq \frac{A}{b}.$$

Similarly, since $g(x)$ is positive and $g(x) \leq B$, we have that $\frac{1}{g(x)} \geq \frac{1}{B}$. Since $f(x) \geq a$ and $a > 0$, we deduce that

$$\frac{f(x)}{g(x)} \geq f(x) \cdot \frac{1}{B} \geq \frac{a}{B}.$$

So (B) must be true. We leave it to the reader to construct counterexamples showing that none of the other inequalities must hold.

Attention! The following 5 questions (#8 – #12) all concern the functions $c(\theta)$ and $s(\theta)$, which are defined as follows. Given an angle θ, which we will measure in degrees, construct a triangle with angles θ and $45°$, and side lengths as indicated in the diagram.

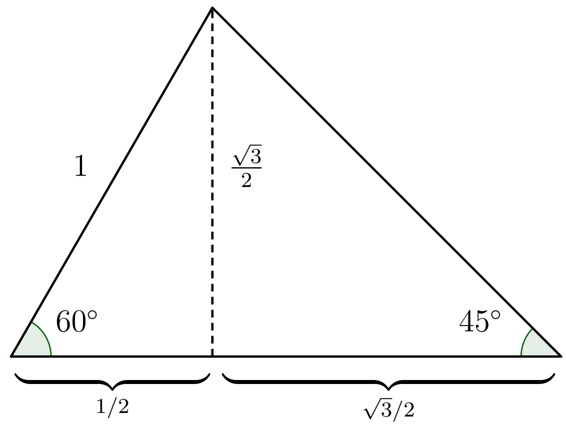


Define $c(\theta) = \dfrac{p}{r}$ and $s(\theta) = \dfrac{q}{r}$.

Problem 8. Find $c(60°)$.

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) 1 (E)$^\heartsuit$ $\frac{1+\sqrt{3}}{2}$

Solution. See the following diagram:



Problem 9. What is the domain of $c(\theta)$?

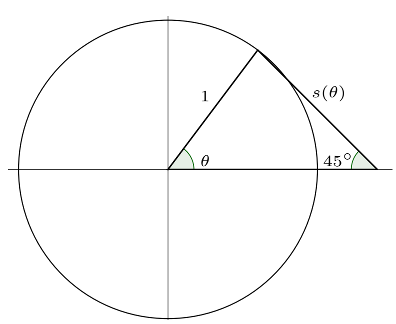
(A) $[0°, 360°)$ (B) $(0°, 360°)$ (C) $(0°, 90°)$ (D)$^\heartsuit$ $(0°, 135°)$ (E) all real numbers

Solution. The angles θ and $45°$ must be part of a triangle whose third angle is $180° - 45° = 135°$. So we must have $0° < \theta < 135°$.

Problem 10. What is the range of $s(\theta)$?

(A) $[-1, 1]$ (B) $[0, 1]$ (C) $(0, 1)$ (D) $(0, \sqrt{2})$ (E)$^\heartsuit$ $(0, \sqrt{2}]$

Solution.



Fix $r = 1$, so that the side opposite the $45°$ angle is the radius of a unit circle. Then $s(\theta)$ is the length of the segment from $(\cos(\theta), \sin(\theta))$ to the positive x-axis, making a $45°$ angle. The largest of these occurs when $\theta = 90°$, when $s(90°) = \sqrt{2}$.

Problem 11. If $c(\theta) = 1$, what is θ?

(A) $0°$ (B) $30°$ (C) $45°$ (D) $60°$ (E)$^\heartsuit$ $90°$

Solution. If $c(\theta) = 1$, then $p = r$, so the triangle is isosceles. The angles opposite p and r are each $45°$, so $\theta = 90°$.

Problem 12. The curve in the plane parametrized by $(c(\theta), s(\theta))$ is part of which kind of curve?

(A) a sine curve (B) a circle (C)$^\heartsuit$ an ellipse that is not a circle (D) a hyperbola (E) a line

Solution. The law of cosines implies that $r^2 = p^2 + q^2 - 2pq\cos(45°)$. Dividing by r^2 gives $1 = c^2 + s^2 - \sqrt{2}cs$. This is the equation of a conic. To see which type of conic, relabel c as x and s as y; then the equation of the conic assumes the "standard form"

$$x^2 - \sqrt{2}xy + y^2 - 1 = 0,$$

with discriminant $(\sqrt{2})^2 - 4 \cdot (1)(1) = -2$. Since the discriminant is negative, the conic is an ellipse. If it were a circle, then the coefficient of xy would vanish. So the equation defines a noncircular ellipse.

Problem 13. Suppose a and b are nonzero decimal digits (1–9), with the property that

$$(aa)^2 + (bb)^2 = aabb.$$

What is $a + b$?

(A) 8 (B) 10 (C)$^\heartsuit$ 11 (D) 16 (E) there are no such numbers

Solution. Since aa and bb are the multiples of 11 by a and b, it follows that

$$11^2 \mid (aa)^2 + (bb)^2 = aabb.$$

Hence, the quotient $aabb/11 = a0b$ possesses a further factor of 11. By a familiar divisibility test,

$$11 \mid a - 0 + b = a + b.$$

Since a and b are nonzero digits, we have $0 < a+b < 18$, and so $a+b = 11$. This shows that the answer is (C), provided that there is such a pair of digits a, b. To check this, note that the only candidates for ab are $29, 38, 47, 56, 65, 74, 83, 92$; looking modulo 10 (i.e., at the final digits of $(aa)^2 + (bb)^2$ and $aabb$), all of these possibilities are quickly eliminated except 83, and in fact

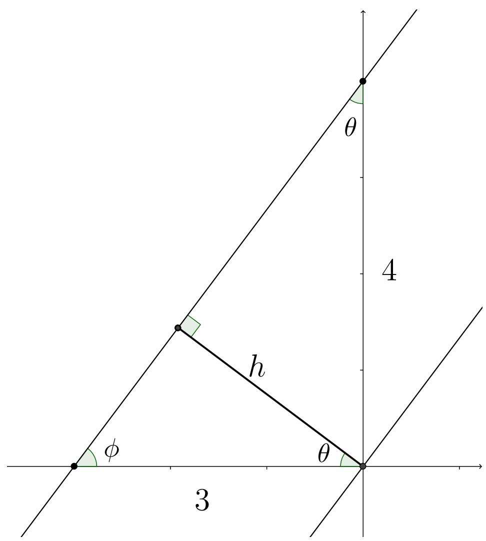
$$88^2 + 33^2 = 8833.$$

So the answer is indeed (C).

Problem 14. a and b are lines in the plane. If you translate a to the left 3 units, it coincides with b. If instead you translate a up by 4 units, it coincides with b again. What is the distance between a and b?

(A) 2 (B)$^\heartsuit$ $\frac{12}{5}$ (C) $\sqrt{\frac{12}{7}}$ (D) 2.5 (E) $\frac{1+\sqrt{5}}{2}$

Solution.



Both lines must have slope 4/3. By translating both lines, we can assume a is described by the equation $y = \frac{4}{3}x$, as shown. We need to find the height h. The two angles marked θ are congruent (since both are complementary to ϕ), and so

$$\sin(\theta) = h/4, \qquad \cos(\theta) = h/3.$$

By the Pythagorean identity,

$$\begin{aligned} 1 &= \sin(\theta)^2 + \cos(\theta)^2 \\ &= h^2\left(\frac{1}{4^2} + \frac{1}{3^2}\right) = \left(\frac{5}{12}\right)^2 h^2, \end{aligned}$$

and so $h = 12/5$.

Problem 15. How many ordered pairs of positive integers (x, y) satisfy $x \cdot y \leq 100$.

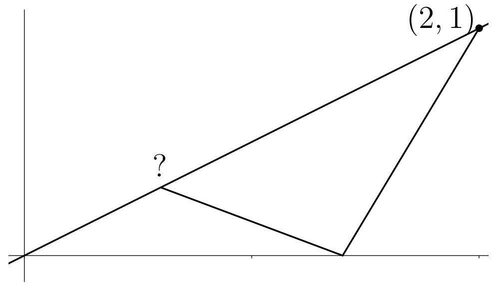
(A) 432 (B)$^\heartsuit$ 482 (C) 532 (D) 572 (E) 582

Solution. If $x \cdot y \leq 100$, then either x or y is at most 10. Put the solutions (x, y) into three categories: (a) those where $x \leq 10$, (b) those where $y \leq 10$, (c) those where both $x, y \leq 10$. Letting A, B, and C be the number of solutions in classes (a), (b), and (c), we see that the total number of solutions is $A + B - C$. (We subtract C to avoid double-counting.) Clearly, $C = 10 \cdot 10^2 = 100$. Equally clearly, $A = B$. To determine A, note that given x, the number of corresponding y with $xy \leq 100$ is precisely $\lfloor 100/x \rfloor$; hence,

$$\begin{aligned} A &= \lfloor 100/1 \rfloor + \lfloor 100/2 \rfloor + \cdots + \lfloor 100/10 \rfloor \\ &= 100 + 50 + 33 + 25 + 20 + 16 + 14 + 12 + 11 + 10 \\ &= 291. \end{aligned}$$

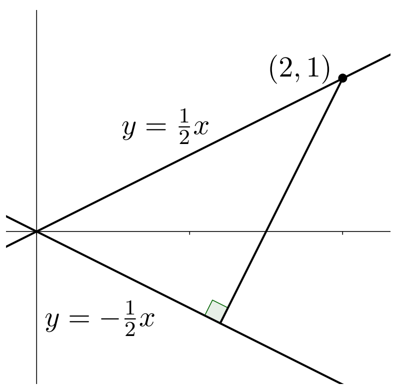
So the total number of solutions is $2 \cdot 291 - 100 = 482$.

Problem 16. What is the length of the shortest path that starts at $(2, 1)$, touches the x-axis, then returns to some point on the line $y = \frac{1}{2}x$.



(A) 1 (B) $\sqrt{5}$ (C) 2 (D)$^\heartsuit$ $\frac{\sqrt{80}}{5}$ (E) $\frac{\sqrt{80}+5}{5}$

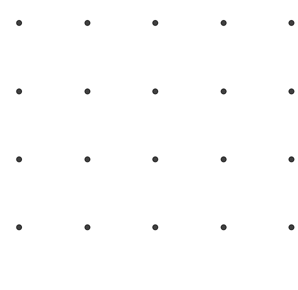
Solution.



Look at the reflection of the line $y = \frac{1}{2}x$ over the x-axis. We need the shortest path from $(2, 1)$ to that line. This will lie on a line perpendicular to $y = -\frac{1}{2}$x, so it lies on the line of slope 2 through $(2, 1)$: $y = 2x - 3$. This intersects $y = -\frac{1}{2}x$ at $(x, y) = (\frac{6}{5}, -\frac{3}{5})$. So the length of the shortest path is

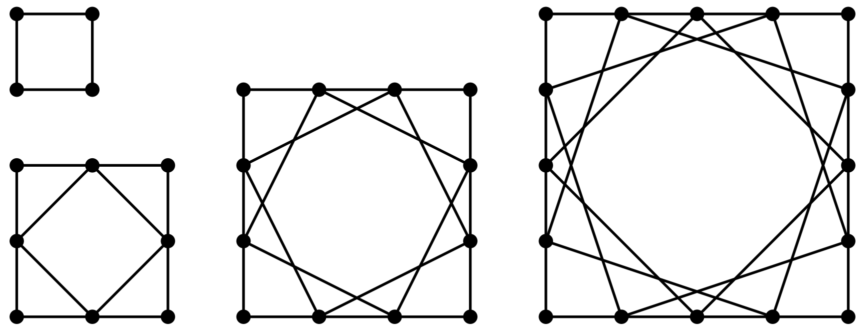
$$\sqrt{\left(2 - \frac{6}{5}\right)^2 + \left(1 - \left(-\frac{3}{5}\right)^2\right)} = \frac{\sqrt{80}}{5}.$$

Problem 17. Consider a 5×5 grid of points, as pictured at right. How many squares can be drawn with all four corners on the grid?



(A) 30 (B) 40 (C) 48 (D) 49 (E)$^\heartsuit$ 50

Solution. Consider a $(k+1) \times (k+1)$ grid of points. There are exactly k squares with corners on the outside edges of this grid.



In an $n \times n$ grid, the number of $(k+1) \times (k+1)$ grids is $(n-k)^2$. Thus, the total number of squares which can be formed is

$$\sum_{k=1}^{4} (5-k)^2 k = 4^2(1) + 3^2(2) + 2^2(3) + 1^2(4) = 50.$$

Problem 18. Let

$$S = 3 + 33 + 333 + \ldots + \underbrace{333 \cdots 333}_{32\ 3\text{'s}}.$$

Find the sum of the digits of S.

(A) 189 (B) 153 (C)$^{\heartsuit}$ 108 (D) 99 (E) 135

Solution. We have

$$\begin{aligned}
S &= 3(1 + 11 + \ldots + 111 \cdots 111) \\
&= 3\left(\frac{10^1 - 1}{9} + \frac{10^2 - 1}{9} + \cdots + \frac{10^{32} - 1}{9}\right) \\
&= \frac{1}{3}\left(10^1 + 10^2 + \cdots + 10^{32} - 32\right) \\
&= \frac{1}{3}\left(\underbrace{111 \cdots 111}_{30\ 1\text{'s}}110 - 32\right) \\
&= \frac{1}{3}\left(\underbrace{111 \cdots 111}_{30\ 1\text{'s}}078\right) \\
&= \underbrace{037037 \cdots 037}_{30\ \text{digits}}026.
\end{aligned}$$

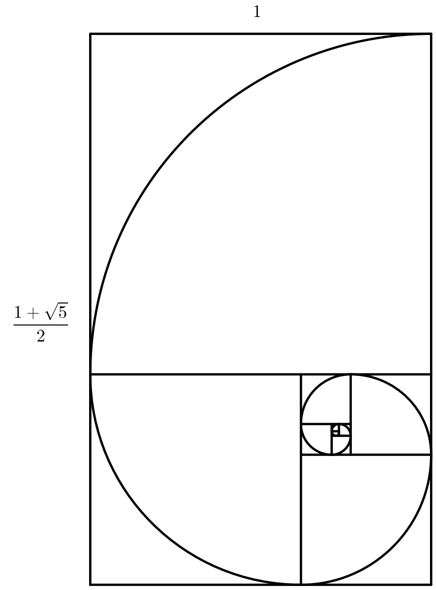
So the sum of the digits of S is $(3+7) \cdot 10 + 2 + 6 = 108$.

Problem 19. Recall that a "golden rectangle" has the following properties:

(1) The ratio of the long side to short side is $g = \frac{1+\sqrt{5}}{2}$.

(2) If you draw a line separating the rectangle into a square and a smaller rectangle, then the smaller rectangle is similar to the original rectangle.

Now draw a quarter circle in each square as shown. How long is this "golden spiral"?



(A) $\dfrac{\pi(1+\sqrt{5})}{4}$ (B)♡ $\dfrac{\pi(3+\sqrt{5})}{4}$ (C) $\dfrac{\pi(1+3\sqrt{5})}{4}$ (D) $\dfrac{\pi(3+3\sqrt{5})}{4}$ (E) infinity

Solution. Since the starting square has length 1, the first arc has length $\frac{2\pi}{4} \cdot 1 = \frac{\pi}{2}$. Each successive square has side length $\frac{1}{g}$ times that of the previous square. Thus, the length of the spiral is given by the following geometric series:

$$\sum_{k=0}^{\infty} \frac{\pi}{2}\left(\frac{1}{g}\right)^k = \frac{\pi}{2}\frac{1}{1-\frac{1}{g}} = \frac{\pi}{2}\frac{g}{g-1} = \frac{\pi}{2}g^2.$$

(We used in the last step that g is a root of $x^2 - x - 1$, so that $g - 1 = 1/g$.) But $g^2 = \frac{3+\sqrt{5}}{2}$, and so $\frac{\pi}{2}g^2 = \pi\frac{3+\sqrt{5}}{4}$.

Problem 20. Find the exact value of $\cos(\pi/7)\cos(2\pi/7)\cos(3\pi/7)$.

(A)♡ $\frac{1}{8}$ (B) $\frac{\sqrt{2}}{8}$ (C) $\frac{\sqrt{3}}{8}$ (D) $\frac{\sqrt{5}}{8}$ (E) $\frac{\sqrt{6}}{8}$

Solution. Let

$$A = \cos(\pi/7)\cos(2\pi/7)\cos(3\pi/7),$$
$$B = \sin(\pi/7)\sin(2\pi/7)\sin(3\pi/7).$$

Then

$$\begin{aligned}
8AB &= 2\cos(\pi/7)\sin(\pi/7) \cdot 2\cos(2\pi/7)\sin(2\pi/7) \cdot 2\cos(3\pi/7)\sin(3\pi/7) \\
&= \sin(2\pi/7)\sin(4\pi/7)\sin(6\pi/7) \\
&= \sin(2\pi/7)\sin(3\pi/7)\sin(\pi/7) \\
&= B.
\end{aligned}$$

(Here we used the double angle formula for sin to go from the first line to the second, and we used the fact that $\sin(\pi - x) = \sin x$ to move from the second to the third.) Since $B \neq 0$, we conclude that $A = \frac{1}{8}$.

Problem 21. Let $a_0 = 2$, $a_1 = 1$, $a_2 = 2$, and for $n \geq 2$ define

$$a_{n+1} = \frac{a_n + a_{n-1} + a_{n-2}}{3}.$$

Find $\lim_{n\to\infty} a_n$.

(A) $\frac{\sqrt{5}-1}{2}$ (B) $\sqrt{2}$ (C) $\frac{3}{2}$ (D) $\frac{1+\sqrt{5}}{2}$ (E)$^\heartsuit$ $\frac{5}{3}$

Solution. We find and use an invariant. Notice that

$$3a_{n+1} + 2a_n + a_{n-1} = 3a_n + 2a_{n-1} + a_{n-2} = \cdots = 3a_2 + 2a_1 + a_0;$$

to see this, multiply the recurrence relation by 3, then add $2a_n + a_{n-1}$ to both sides. When n is very large, a_{n+1}, a_n, and a_{n-1} all approach the same limit. Thus, writing lim for the limit as $n \to \infty$,

$$\begin{aligned} 6 \lim a_n = \lim 6a_n &= \lim(3a_{n+1} + 2a_n + a_{n-1}) \\ &= \lim(3a_2 + 2a_1 + a_0) = 3a_2 + 2a_1 + a_0 = 3 \cdot 2 + 2 \cdot 1 + 2 = 10. \end{aligned}$$

Hence, $\lim a_n = \frac{10}{6} = \frac{5}{3}$.

Problem 22. Two children are playing on two toy pianos. Each toy piano has 5 notes. Every second each child switches at random from hitting the current note to a different but neighboring note. If the children start at random notes, what is the probability that they will eventually play the same note at the same time?

(A) $\frac{12}{25}$ (B)$^\heartsuit$ $\frac{13}{25}$ (C) $\frac{3}{5}$ (D) $\frac{4}{5}$ (E) 1

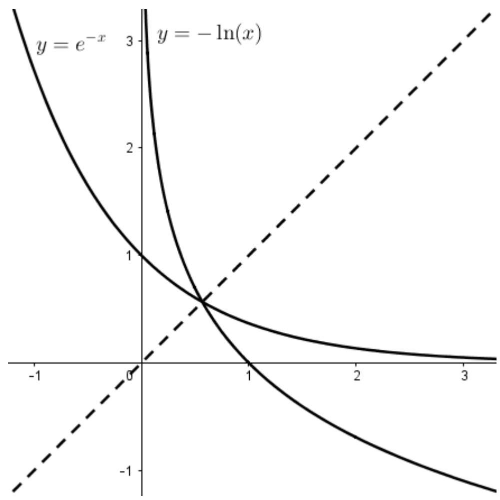
Solution. Consider the difference between the two childrens' notes. When they switch notes, the difference between the notes remains even if it was even before and remains odd if it was odd before. Thus if they start at notes which are an odd distance away, they will never play the same note. On the other hand, eventually one child will switch from playing a note lower than the other child's to playing a higher note than the other child's (with probability 1). If the difference between the two childrens' notes was even, then this change must include having played the same note. Thus the probability is just that of starting an even distance away from each other, i.e.

$$\left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{13}{25}.$$

Problem 23. The functions e^{1-x} and $-\ln(x-1)$ intersect only at the point (p,q). What is $q-p$?

(A) $-\ln 2$ (B) $-\frac{e}{3}$ (C)$^{\heartsuit}$ -1 (D) $-\frac{4}{3}$ (E) $-\frac{e}{2}$

Solution. Notice that e^{-x} and $-\ln(x)$ are inverse functions i.e. if $e^{-a}=b$, then $-\ln(b)=a$. Thus if (a,b) is on the graph of e^{-x}, then (b,a) is on the graph of $-\ln(x)$. Another way of saying this is that the graph of e^{-x} is the graph of $-\ln(x)$ reflected over the line $y=x$. Therefore if these two functions intersect at only one point, it must be on the line $y=x$.



The given functions $e^{-(x-1)}$ and $-\ln(x-1)$ are merely e^{-x} and $-\ln(x)$ shifted one unit to the right. Thus the point of intersection lies on the line $y=x-1$ giving $q-p=-1$.

Problem 24. For how many positive integers $n \le 5^5$ does 5 not divide $\dbinom{2n}{n}$?

(A)$^{\heartsuit}$ 243 (B) 256 (C) 625 (D) 2401 (E) 2500

Solution. The exponent on the power of 5 dividing $m!$ is given by $\lfloor m/5\rfloor + \lfloor m/5^2\rfloor + \dots$. Hence, the highest power of 5 dividing $\binom{2n}{n} = \frac{(2n)!}{n!n!}$ is

$$\sum_{k=1}^{\infty}\left(\left\lfloor \frac{2n}{5^k}\right\rfloor - 2\left\lfloor \frac{n}{5^k}\right\rfloor\right).$$

Let $\{t\}$ denote the fractional part of the real number t, defined by $\{t\} = t - \lfloor t\rfloor$. Then

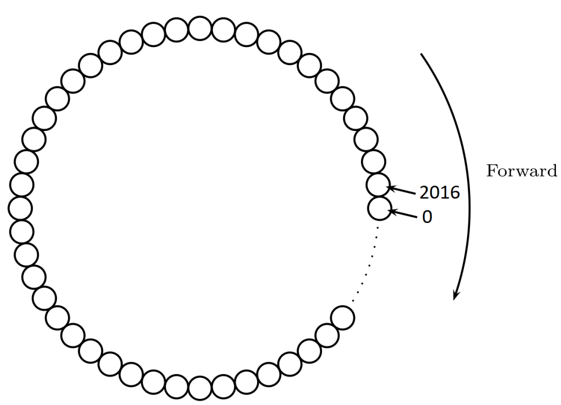
$$\lfloor 2t\rfloor - 2\lfloor t\rfloor = (2t - \{2t\}) - 2(t - \{t\}) = 2\{t\} - \{2t\}.$$

This last quantity is always 0 or 1, according to whether $0 \le \{t\} < 1/2$ or $\{t\} \ge 1/2$, respectively. In particular, the sum on k is a sum of nonnegative terms, and so is zero precisely when each of its terms is zero. Moreover, if $n = a_j a_{j-1} a_{j-2} \dots a_0$ in base 5 (so that each digit $a_i \in \{0, 1, 2, 3, 4\}$), then

$$\{n/5^k\} = 0.a_{k-1}a_{k-2}\dots a_0,$$

which is greater than or equal to $1/2$ precisely when $a_{k-1} = 3$ or 4. Hence, the above sum on k vanishes if and only if n has no 3's or 4's in its base 5 expansion. So below each power 5^K, there are 3^K nonnegative integers n for which $5 \nmid \binom{2n}{n}$. We take $K = 5$. Now in our problem, we we count up to and *including* 5^5, which adds 1 to the count, but we exclude 0, which takes 1 away— so it's a wash, and the final answer is $3^5 = 243$.

Problem 25. A circular arrangement of 2017 lily pads floats in Shifrin pond. The lily pads are consecutively numbered 0 to 2016, with lily pad 2016 next to lily pad 0. Ted the Toad can jump forward n^2 steps, for any positive integer n, wrapping around the circle as necessary. (For example, starting from lily pad 0, he can reach lily pad 8 in one jump of length $45^2 = 2025$.) What is the smallest positive integer d such that, starting from lily pad 0, Ted can reach any other lily pad within d jumps?



(A)$^\heartsuit$ 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution. This is equivalent to finding the smallest d for which every integer is congruent, modulo 2017, to a sum of d squares. Let $p = 2017$ and note that p is a prime number. Thus, if $a^2 \equiv b^2 \pmod{p}$, then

$$p \mid a^2 - b^2 = (a-b)(a+b),$$

and so either $a \equiv b$ or $a \equiv -b \pmod{p}$. It follows that there are exactly 1009 distinct squares modulo 2017, namely, the squares $0^2, 1^2, 2^2, \dots, (\frac{p-1}{2})^2 = 1008^2$. In particular, not everything is a square, and so $d > 1$. We claim that everything is representable as a sum of two squares modulo 2017, so that $d = 2$. To see this, let m be any integer, and consider the list $m - 0^2, m - 1^2, \dots, m - 1008^2$. All these numbers are distinct modulo 2017. Since $1009 + 1009 > 2017$, no two 1009-element lists of numbers can be disjoint modulo 2017. Thus, there are integers a, b with $0 \le a, b \le 1008$ with

$m - a^2 \equiv b^2 \pmod{2017}$. But then $m \equiv a^2 + b^2 \pmod{2017}$, so that m is congruent to a sum of two squares.

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