2019 GRADUATE PRELIMINARY EXAM

All problems are weighted equally. Throughout $\mathbb{R}$ denotes the real numbers.

(1) Negate each of the following sentences. (Do not simply say "It is not the case that...").
   (a) The integer x is odd, the integer y is odd and the integer z is even.
   (b) Every prime number is odd.
   (c) Either you pay the bill or we're not going to dinner.
   (d) If I'm lying, I'm dying.

(2) Let $f(x) = e^{x^2}$. Show: for every positive integer n, there is a degree n polynomial $P_n(x)$ such that for all $x \in \mathbb{R}$ we have $f^{(n)}(x) = P_n(x)e^{x^2}$. (Here $f^{(n)}$ denotes the nth derivative of f.)

(3) Let X and Y be nonempty sets.
   (a) Let $f : X \to Y$ and $g : Y \to X$ be functions. Show: if $g \circ f$ is injective, then f is injective.
   (b) Let $f : X \to Y$ and $g : Y \to X$ be functions. Show: if $g \circ f$ is surjective, then g is surjective.
   (c) Let $g : Y \to X$ be a surjective function. Show: there is a function $f : X \to Y$ such that $g \circ f = 1_X$, i.e., for all $x \in X$ we have $g(f(x)) = x$.
   (d) Let $f : X \to Y$ be an injective function. Show: there is a function $g : Y \to X$ such that $g \circ f = 1_X$.

(4) For each part, either find a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the given properties or show that no such linear transformation exists.
   (a) The kernel and image of L are both spanned by $(1, 2, 3)$.
   (b) The kernel of L is spanned by $(1, 2, 3)$ and the image of L is $\{(x, y, z) \mid 3x + 4y + 5z = 0\}$.

(5) Diagonalize the matrix $M = \begin{bmatrix} 1 & -2 \\ 4 & 7 \end{bmatrix}$ over $\mathbb{R}$.

(6) Show directly from the ϵ, δ definition that

$$\lim_{x \to 3} \frac{1}{x-2} = 1.$$

(7) Let $I \subset \mathbb{R}$ be an interval, and let $\{f_n : I \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of real-valued functions.
   (a) Give a careful definition of: "The sequence $\{f_n\}$ converges pointwise on I."
   (b) Give a careful definition of: "The sequence $\{f_n\}$ converges uniformly on I."
   (c) Let $f_n = x^n$. Show: the sequence $\{f_n\}$ converges pointwise on $[0, 1]$ and does not converge uniformly on $[0, 1]$.

(8) (a) Let G be a finite commutative group, written multiplicatively, with identity element e. Suppose that G has exactly one element, t of order 2 – i.e., such that $t^2 = e$ and $t \neq e$. Show: $\prod_{x \in G} x = t$. (Hint: for all $x \in G$ we have $xx^{-1} = e$.)
   (b) Let p be a prime number. Deduce from part a) that $(p-1)! \equiv -1 \pmod{p}$.