Ph.D. Qualifying Exam, Probability Theory, August 2006

(Solve any 5 problems.)

1. Let $\{X_n\}$ be a sequence of independent random variables.

   (a) If $EX_n = 0$ for $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} \text{var}(X_n) < \infty$, show that $\sum_{n=1}^{\infty} X_n$ converges a.s.

   (b) State (without proof) Levy's inequality and use it to prove that $S_n = \sum_{k=1}^{n} X_k$ converges a.s. if and only if it converges in probability.

2. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteristic functions.

   (b) Let $\{X_n\}$ be iid r.v.s with distribution $F(x)$ having finite mean μ and varivance σ^2. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

3. (a) Given a random variable X with finite mean square. Let $\mathcal{D}$ be a σ-algebra. Show that $E[X|\mathcal{D}]$ is the minimizer of $E(X-\xi)^2$ over all $\mathcal{D}$-measurable r.v.s ξ, i.e.,

$$E(X - E[X|\mathcal{D}])^2 \le E(X - \xi)^2$$

   for all $\mathcal{D}$-measurable r.v.s ξ.

   (b) Let $(\Omega, \mathcal{F}, P)$ denote a probability space. Suppose $f: R^n \times \Omega \to R$ is a bounded $\mathcal{B}(R^n) \times \mathcal{C}$ measurable function and X be a R^n-valued $\mathcal{D}$ measurable random variable. Assume $\mathcal{C}$ and $\mathcal{D}$ are independent. If $g(x) := Ef(x,\omega)$, then

$$g(X) = E[f(X,\omega)|\mathcal{D}], \text{ a.s.}$$

4. (a) Let $(\Omega, \mathcal{F}, P)$ be the probability space with $\Omega = [0,1]$, $\mathcal{F} =$ the Borel σ-algebra on $[0,1]$, and $P =$ Lebesgue measure on $\mathcal{F}$. Let

$$X_n = \frac{n}{\log n} I_{(0,\frac{1}{n})}.$$

   Show that $\{X_n : n \ge 2\}$ is uniformly integrable.

   (b) Show that for any two random variables X and Y with $Var(X) < \infty$,

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)].$$

5. Let $\{X_n\}$ be iid r.v.s. Then,

   (a) $n^{-1} \max_{1 \le i \le n} |X_i| \to 0$ in probability iff $nP(|X_1| > n) = o(1)$.

   (b) $n^{-1} \max_{1 \le i \le n} |X_i| \to 0$ a.s. iff $E|X_1| < \infty$.

6. Suppose that X and Y are independent random variables with the same exponential density

$$f(x) = \theta e^{-\theta x}, \ x > 0,$$

   with $\theta > 0$. Show that the sum $X + Y$ and the ratio X/Y are independent.

7. (a) Let X_t be an $\mathcal{F}_t$–martingale and ϕ a convex function with $E|\phi(X_t)| < \infty$ for all $t \ge 0$. Show that $\phi(X_t)$ is an $\mathcal{F}_t$–submartingale.

   (b) Let X_t be a submartingale. Show that $\sup_t E|X_t| < \infty$ iff $\sup_t E(X_t)^+ < \infty$.