Probability Theory, Ph.D Qualifying, Spring 2012

Completely solve any six problems.

1. If the independent random variables $\{X_n\}$ satisfy the condition
$$Var(X_i) \le c < \infty, \ i = 1, 2, \dots,$$
show that the SLLN holds.

2. (a) Let $(\Omega, \mathcal{F}, P)$ be a probability space. When do you say that $\mathcal{F}$ is P-trivial?
(b) Show that a σ-field is P-trivial if and only if $\mathcal{F}$ is independent of itself. Also show in this case, that any $\mathcal{F}$-random variable is constant almost surely.

3. Prove for iid random variables $\{X_n\}$ with $S_n = X_1 + \cdots + X_n$ that
$$\frac{S_n - C_n}{n} \to 0 \text{ a.s.}$$
for some sequence of constants C_n if and only if $E|X_1| < \infty$.

4. Let $X_n, \ n \ge 1$, be a sequence of i.i.d nondegenerate real random variables, and put $S_n := X_1 + \cdots + X_n, \ n \ge 1$. Show that:
(a) $P(S_n \in B \ i.o) = 0$ or 1, for any $B \in \mathcal{B}(\mathbb{R})$;
(b) $\limsup_n S_n = \infty \ a.s \ or -\infty \ a.s$;
(c) $\limsup_n (\pm S_n) = \infty \ a.s$ if the X_n are symmetric.

5. Let $X, \{X_n, \ n \ge 1\}, \{Y^{(k)}, \ k \ge 1\}, \{Y_n^{(k)}, \ n \ge 1, \ k \ge 1\}$, be real random variables such that $Y_n^{(k)} \longrightarrow Y^{(k)}$ in distribution as $n \to \infty$, for fixed k, and that $Y^{(k)} \to X$ in distribution as $k \to \infty$. Show that $X_n \to X$ in distribution if $\lim_{k\to\infty} \limsup_{n\to\infty} E[|Y_n^{(k)} - X_n| \wedge 1] = 0$.

6. (a) Define the uniform integrability of a family $X_t, \ t \in T$, of random variables.
(b) Let $X \in L^1(\Omega, \mathcal{F}, P)$, and $\mathcal{A}_t, \ t \in T$ be a family of subσ-fields of $\mathcal{F}$, where T is an index set. Show that the conditional expectations $\{E[X| \ \mathcal{A}_t], \ t \in T\}$ form a uniformly integrable family.

7. Let $\{X_n, \ n \ge 1\}$ be a sequence of independent identically distributed random variables with $E|X_1| < \infty$. Show that
$$\lim_{n\to\infty} \frac{1}{n} E(\max_{1\le k\le n} |X_k|) = 0.$$

8. Let $\{X_n\}$ be iid r.v.s with $E|X_1| < \infty$. Show that $\sum (-1)^n X_n/n$ converges a.s.

9. (a) Given a discrete-time filtration $(\Omega, \mathcal{F}, \mathcal{F}_n, n \ge 1)$, define a process X_n predictable w.r.t to the filtration.
(b) Let $\mathcal{F}_n, \ n \ge 1$, be a filtration on $(\Omega, \mathcal{F}, P)$ and $X_n, \mathcal{F}_n, \ n \ge 1$, be any integrable adapted process.
i. Show that X has an a.s unique decomposition $X_n = M_n + A_n, \ n \ge 1$, such that $\{M_n, \mathcal{F}_n\}$ is a martingale and $\{A_n, \mathcal{F}_n\}$ is a predictable process with $A_0 = 0$.
ii. Show that X_n is a submartingale if and only if A_n is a.s nondecreasing (in n.)