Probability Theory, Ph.D Qualifying, Spring 2019

Completely solve any five problems.

1. (a) Show that the mean μ of a random variable X has the property

$$\min_c E(X-c)^2 = E(X-\mu)^2 = V(X).$$

(b) Prove that for any r.v. X

$$E|X| = \int_0^\infty P(|X| \geq t)dt.$$

2. Suppose that X and Y are independent random variables with the same exponential density

$$f(x) = \theta e^{-\theta x},\ x > 0.$$

Show that the sum $X+Y$ and the ratio X/Y are independent.

3. Given a square integrable r.v. X, show that for $\lambda \geq 0$,

$$P(X - EX \geq \lambda) \leq \frac{\sigma^2(X)}{\sigma^2(X) + \lambda^2}.$$

4. (a) Given a random variable X with finite mean square. Let $\mathcal{D}$ be a σ-algebra. Show that $E[X|\mathcal{D}]$ is the minimizer of $E(X-\xi)^2$ over all $\mathcal{D}$-measurable r.v.s ξ, i.e.,

$$E(X - E[X|\mathcal{D}])^2 \leq E(X-\xi)^2$$

for all $\mathcal{D}$-measurable r.v.s ξ.

(b) Let $(\Omega, \mathcal{F}, P)$ denote a probability space. Suppose $f:\ R^n \times \Omega \to R$ is a bounded $\mathcal{B}(R^n) \times \mathcal{C}$ measurable function and X be a n-dimensional $\mathcal{D}$ measurable random variable. Assume $\mathcal{C}$ and $\mathcal{D}$ are independent. If $g(x) := Ef(x,\omega)$, then

$$g(X) = E[f(X,\omega)|\mathcal{D}],\ \text{a.s.}$$

5. Let $\{X_n,\ n \geq 1\}$ be a sequence of independent identically distributed random variables with $E|X_1| < \infty$. Show that

$$\lim_{n\to\infty} \frac{1}{n} E(\max_{1\leq k\leq n} |X_k|) = 0.$$

6. Let $\{X_n\}$ be iid r.v.s. Then,

(a) $n^{-1}\max_{1\leq i\leq n}|X_i| \to 0$ in probability if and only if $nP(|X_1| > n) = o(1)$.

(b) $n^{-1}\max_{1\leq i\leq n}|X_i| \to 0$ a.s. if and only if $E|X_1| < \infty$.

7. Let $X_1, X_2, \ldots$ be a sequence of independent r.v.s with $EX_i = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Show that $\phi(S_n)$ is an $\mathcal{F}_n$–submartingale for any convex ϕ provided that $E|\phi(S_n)| < \infty$ for all n.