PDE Qualifying Exam — Syllabus

One-semester graduate course · *Evans* PDE *(Ch. 1–5)* · *Shubin* Lectures on PDE *(Ch. 1–5)*

I. First-Order PDEs and Characteristics (Evans Ch. 3)

- Linear, semilinear, quasilinear, and fully nonlinear equations
- Characteristic ODE system: $\dot{x} = D_p F$, $\dot{z} = D_p F \cdot p$, $\dot{p} = -D_x F - D_u F \cdot p$
- Local existence theorem (non-characteristic condition)
- Method of characteristics for the Cauchy problem
- Burgers' equation: shock formation and breakdown time $T^* = -1/\min u_0'$
- Rankine–Hugoniot jump condition

II. Classical Linear PDEs (Evans Ch. 2)

Laplace / Poisson equation $(-\Delta u = f)$

- Fundamental solution $\Phi(x)$; representation formula $u = \Phi * f$
- Mean-value property (characterizes harmonic functions)
- Strong maximum principle and Liouville's theorem
- Green's function; Poisson kernel on ball and half-space
- Method of images

Heat equation $(u_t - \Delta u = 0)$

- Heat kernel $\Phi(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$
- Cauchy problem; infinite speed of propagation
- Parabolic maximum principle
- Duhamel's principle for $u_t - \Delta u = f$

Wave equation $(u_{tt} - \Delta u = 0)$

- d'Alembert's formula (1D)
- Kirchhoff's formula (3D); method of descent (2D)
- Finite speed of propagation; domain of dependence
- Huygens' principle $(n = 3)$; energy conservation
- Duhamel's principle for $u_{tt} - \Delta u = f$

III. Distributions and Fourier Analysis (Shubin Ch. 1–5)

- Test function spaces $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$; topology and convergence
- Distributions $\mathcal{D}'(\Omega)$: regular functionals, delta distribution
- Differentiation of distributions; multiplication by C^∞ functions
- Convolution of distributions; fundamental solutions $(LE = \delta_0)$
- Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and tempered distributions $\mathcal{S}'$
- Fourier transform on $\mathcal{S}$: derivative rule, Plancherel, convolution theorem, inversion formula
- Extension of $\mathcal{F}$ to $\mathcal{S}'$; Fourier characterization of $H^s(\mathbb{R}^n)$
- Solving constant-coefficient PDEs via $\mathcal{F}$ (heat kernel derivation)

IV. Sobolev Spaces $W^{k,p}(\Omega)$ (Evans Ch. 5)

- Weak derivatives; definition and properties of $W^{k,p}(\Omega)$ and $H^k(\Omega)$
- $W_0^{k,p}(\Omega)$; density of $C^\infty(\bar{\Omega})$
- Extension theorem; trace theorem $(Tu = u|_{\partial\Omega})$
- **Sobolev inequalities:**
  - GNS $(1 \le p < n)$: $\|u\|_{L^{p^*}} \le C\|Du\|_{L^p}$, $p^* = \frac{np}{n-p}$
  - Morrey $(p > n)$: $|u(x) - u(y)| \le C|x-y|^{1-n/p}\|Du\|_{L^p}$
  - General embedding: $W^{k,p} \hookrightarrow L^q$ or $W^{k,p} \hookrightarrow C^{m,\gamma}$
- Rellich–Kondrachov compact embedding $(W^{1,p} \hookrightarrow\hookrightarrow L^q,\ q < p^*)$
- Poincaré inequality: $\|u\|_{L^p} \le C\|Du\|_{L^p}$ on $W_0^{1,p}(\Omega)$

Key Theorems and Proof Techniques

1. Fundamental solution of $-\Delta$ (spherical symmetry, ODE reduction)
2. Mean-value property $\Leftrightarrow$ harmonic (differentiate average, Green's theorem)
3. Strong max. principle (propagation via MVP; Hopf boundary lemma)
4. Liouville's theorem (consequence of max. principle or mean-value)
5. Heat kernel via Fourier $(\hat{u}_t + |\xi|^2 \hat{u} = 0 \Rightarrow u = \Phi * g)$
6. Parabolic max. principle (auxiliary function $e^{-\lambda t} u$)
7. d'Alembert / Kirchhoff formulas (explicit calculation / spherical means)
8. Finite speed of propagation (energy on shrinking cone)
9. GNS inequality (product trick, generalized Hölder)
10. Poincaré inequality (contradiction + Rellich–Kondrachov)
11. Rellich–Kondrachov (mollification + Arzelà–Ascoli + diagonal argument)
12. Trace theorem (density of $C^1(\bar{\Omega})$, boundary integral estimate)

Exam Calculation Skills • Characteristics & shock time • Green's function / Poisson kernel • Duhamel's principle • Fourier transform for PDE • Membership in $W^{k,p}$; weak derivatives • Sobolev embedding & regularity • Compactness via Rellich–Kondrachov • Uniqueness via max. principle