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Mathematics Department
The University of Georgia

Real Analysis Qualifying Examination

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Give clear reasoning. State clearly which theorem you are using. You should not cite anything else such as examples, exercises, or problems. Cross out the parts you do not want to be graded. Read through all the problems, do them in any order, the one you feel most confident about first. They are not in the order of difficulty.

Notation: $\mathbb{R}$ is the set of real numbers; m is the Lebesgue measure.

Problem #	Points	Grade
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

Committee Recommendation

Grader's Remarks

1. For $n \in \mathbb{N}$, let $e_n = (1 + \frac{1}{n})^n$ and $E_n = (1 + \frac{1}{n})^{n+1}$. It is obvious that $e_n < E_n$. Prove Bernoulli's inequality:

$(1 + x)^n \geq 1 + nx$ for $-1 < x < \infty$ and $n \in \mathbb{N}$.

Then use Bernoulli's inequality or any other method to show that

(a) The sequence e_n is increasing;

(b) The sequence E_n is decreasing;

(c) $2 \leq e_n < E_n \leq 4$;

(d) $\lim_{n\to\infty} e_n = \lim_{n\to\infty} E_n$.

2. Choose $0 < \lambda < 1$ and construct the Cantor set C_λ as follows: Remove from $[0, 1]$ its open middle part of length λ; we left with two intervals I_{11} and I_{12} of equal length. Remove from each of them their open middle parts of length $\lambda m(I_{11})$, etc. and keep doing this *ad infinitum*. We are left with the set C_λ. Prove that the set C_λ has Lebesgue measure zero.

3. Let f be a (Lebesgue) measurable function on $\mathbb{R}$ and E be a measurable subset of $\mathbb{R}$ such that $0 < A = \int_E f(x)\, dx < \infty$. Show that for every number $0 < t < 1$, there exists a measurable subset $E_t \subset E$ such that $\int_{E_t} f(x)\, dx = tA$.

4. Let E be a Lebesgue measurable set in $\mathbb{R}$ with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that (a) f is in $L^1(\mathbb{R})$; (b) f is uniformly continuous; (c) $\lim_{|x| \to \infty} f(x) = 0$.

Hint You may use this identity of characteristic functions: $\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y - x)$.

5. Let $(X, \mathcal{M}, \mu)$ be a measured space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\phi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\}).$$

Show that the function ϕ and ψ are Borel measurable and that

$$\int_X |f| d\mu = \int_0^\infty [\phi(\lambda) + \psi(\lambda)] d\lambda.$$

6. Compute the following without using the Riesz representation theorem:

$$\sup\left\{\left|\int_0^1 f(x)e^x\,dx\right| \;:\; f \in L^2([0,1],m),\, \|f\|_2 \leq 1\right\}$$