Real Analysis Qualifying Exam

August 2007

Work 5 of the following problems; if you try all 6, be sure to indicate which one you don't want counted. Justify all work.

m stands for Lebesgue measure on $\mathbb{R}$.

1. Suppose $f : \mathbb{Q} \to \mathbb{Q}$ is uniformly continuous. Prove that there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(x) = f(x)$ for each $x \in \mathbb{Q}$. Then give an example to explain why *uniform* continuity of f is assumed.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded (above and below) Lebesgue measurable function. Prove that the set $S := \{a \in \mathbb{R} : m(f^{-1}(a, \infty)) = 0\}$ has a smallest member.

3. Evaluate $\lim_{y\to 0} \int_0^1 \frac{1-\exp(y\sqrt{x})}{y}\, dx$, being sure to justify your procedure completely.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that the graph of f is a Lebesgue measurable subset of $\mathbb{R}^2$ with zero measure.

5. Let $(X, \|\ \|)$ be a normed linear space over $\mathbb{R}$ and suppose $\phi : X \to \mathbb{R}$ is a discontinuous linear functional. Prove that there is a sequence (x_n) in X satisfying $\lim_{n\to\infty} \|x_n\| = 0$, but $\phi(x_n) = 1$ for all n. Conclude that the null space of ϕ is not closed.

6. For each open subset U of $\mathbb{R}$, and each $p \in [1, \infty)$, write

$$L^p(U) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ is Lebesgue measurable and } \int_U |f|^p < \infty\}.$$

a) Prove that if U has finite Lebesgue measure, then $L^2(U) \subset L^1(U)$.
b) Prove the converse of Part a).