QUALIFYING EXAMINATION IN REAL ANALYSIS

August 13, 2010

1:30–3:30 pm

The six problems are weighted equally. A^c denotes the complement of the set A, and m denotes Lebesgue measure.

1. For small values of $|x|$, which is larger, $\sin^2 x$ or $\sin(x^2)$?

2. Let $f \in L^1([0,1])$, and for $x \in [0,1]$ define
$$g(x) = \int_x^1 \frac{f(t)}{t}\,dt.$$
Show that $g \in L^1([0,1])$ and that $\displaystyle\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx$.

3. Let $A \subset [0,1]$ be measurable, and define $L^2(A) \subset L^2([0,1])$ to be the subspace consisting of all $f \in L^2$ such that $f \equiv 0$ a.e. off A. Show that
$$L^2(A)^\perp = L^2(A^c)\,.$$

4. Let $f_n \to f$ pointwise a.e. on $[0,1]$. Suppose
$$\limsup_{n\to\infty} \|f_n\|_1 \le \|f\|_1 < \infty.$$
Show that $f_n \to f$ in L^1, i.e., that $\displaystyle\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

5. Suppose $f \in L^1(\mathbb{R})$ and $f \ge 0$. For $y > 0$, let
$$g(y) = m\left(\{x \in \mathbb{R} : f(x) \ge y\}\right)\,.$$
Set $G(y) = yg(y)$.
   a. Prove that $\displaystyle\lim_{y\to 0^+} G(y) = \lim_{y\to\infty} G(y) = 0$ and that G is bounded.
   b. Prove that G achieves its maximum at some point y_0.

6. Let $f_1, f_2, \ldots : \mathbb{N} \to \mathbb{R}$ be a sequence of functions such that $|f_i(n)| \le n$ for all $i, n \in \mathbb{N}$. Show that there is a subsequence $f_{i'}$ converging pointwise on $\mathbb{N}$ to a function f_0.