Real Analysis Qualifying Examination
January 2009

There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let $f : [0,1] \to \mathbb{R}$ be continuous. Evaluate

$$\lim_{k\to\infty} \int_0^1 kx^{k-1} f(x)\,dx$$

2. Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$. Let $\mu : \mathcal{B} \to [0,\infty)$ be a set function with the property that if $\{E_k\}$ is a countable collection of disjoint sets in $\mathcal{B}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

(a) Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supset F_{k+1}$ for all k, then

$$\lim_{k\to\infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

(b) Suppose that for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$, it follows that $\mu(E) = 0$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

3. Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \le M$ for all $k \in \mathbb{N}$. Prove that if $f_k \to f$ almost everywhere, then

$$\lim_{k\to\infty} \int_0^1 f_k(x)\,dx = \int_0^1 f(x)\,dx$$

4. (a) Show that if $f \in L^4([0,1])$, then $f \in L^2([0,1])$ and $\|f\|_2 \le \|f\|_4$.

(b) Does there exist a constant C so that for all $f \in L^4([0,1])$, $\|f\|_4 \le C\|f\|_2$? Justify your answer.

(c) For a fixed function $g \in L^4([0,1])$, let A denote the ratio $\|g\|_4/\|g\|_2$. Find a constant B, depending only on A, such that $\|g\|_2 \le B\|g\|_1$.

5. Let $\mathcal{H}$ be a Hilbert space and $\{\varphi_k\}_{k=1}^{\infty}$ be a subset $\mathcal{H}$ with the property that for every $f \in \mathcal{H}$

$$\sum_{k=1}^{\infty} |\langle f, \varphi_k\rangle|^2 = \|f\|^2$$

where $\langle f, \varphi_k\rangle$ denotes the inner product of f and φ_k in $\mathcal{H}$ and $\|f\|$ denotes the norm of f on $\mathcal{H}$.

(a) Show that for all f, g in $\mathcal{H}$,

$$\langle f, g\rangle = \sum_{k=1}^{\infty} \langle f, \varphi_k\rangle \overline{\langle g, \varphi_k\rangle}$$

[*Hint: Consider* $\|f+g\|^2$ *and* $\|f+ig\|^2$]

(b) Show that for all f in $\mathcal{H}$,

$$\lim_{N\to\infty} \left\| f - \sum_{k=1}^{N} \langle f, \varphi_k\rangle \varphi_k \right\| = 0$$

[*Hint: Use the fact that* $\|h\| = \sup_{\|g\|\le 1} |\langle h, g\rangle|$ *for all* h *in* $\mathcal{H}$]