PRINT NAME: ______________________________
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Real Analysis Qualifying Exam
January 6, 2010

Give clear reasoning. State clearly the theorems you are using. Cross out the parts you do not want to be graded. Read through all the problems, do them in any order, the one you feel most confident about first. They are not necessarily arranged in order of difficulty. You should not cite anything else: examples, exercises, or problems.

Problem #	Points	Score
1	16	
2	16	
3	16	
4	16	
5a	10	
5b	10	
6	16	
Total	100	

Committee Recommendation

Grader's Remarks

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Notation: $\mathbb{R}$ is the set of real numbers; m and m^* are Lebesgue measure and Lebesgue outer measure, respectively.

1. Let f be a polynomial with real coefficients. Show that there exists $x_0 \in \mathbb{R}$ such that $|f(x_0)| \leq |f(x)|$ for all $x \in \mathbb{R}$.

2. Prove that there exists $\epsilon > 0$ such that

$$\cos(\sin x) > \frac{1}{2}(1 + \cos^2 x)$$

whenever $0 < |x| < \epsilon$.

3. Let $f(x) = x^3$ and $E \subset \mathbb{R}$. Show that $m^*(E) = 0$ if and only if $m^*(f(E)) = 0$.

4. Let $f(x)$ be Lebesgue measurable on $\mathbb{R}$. Suppose that $f(x)$ and $xf(x)$ are both integrable over the whole of $\mathbb{R}$. Define

$$F(t) := \int_{\mathbb{R}} f(x)\ \sin(tx)\ dx$$

Prove that $F(t)$ is differentiable at every t, and find a formula for $F'(t)$.

5. Let $f \in L^1(\mathbb{R})$, and for $r > 0$ put

$$f_r(x) := \frac{1}{2r} \int_{x-r}^{x+r} f\, dm.$$

a. Prove that $\| f_r \|_1 \leq \| f \|_1$ for all $r > 0$. (Hint: use Fubini's theorem.)
b. Prove that $f_r \to f$ in L^1 as $r \downarrow 0$.

6. Let $f : [0, 1] \to \mathbb{R}$ be a bounded Lebesgue measurable function. Prove that

$$\lim_{p \to \infty} \| f \|_p = \| f \|_\infty$$