Real Analysis Qualifying Examination

January 2011

There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

Notation: In Questions 3 and 4 below m stands for Lebesgue measure on $\mathbb{R}$.

1. (a) Let $A \subseteq \mathbb{R}^n$. Give a precise definition of what it means for a function $f : A \to \mathbb{R}$ to be uniformly continuous on A.

   (b) Let K be a compact subset of $\mathbb{R}^n$. Prove that if $f : K \to \mathbb{R}$ is continuous on K, then f is uniformly continuous on K.

2. Let $0 < p < \infty$ and $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of complex numbers. Prove the equivalence of the following two statements:

   (i) $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N} |x_n|^p = 0$

   (ii) For every $\varepsilon > 0$, $\lim\limits_{N\to\infty} \frac{1}{N} |\{1 \leq n \leq N : |x_n| \geq \varepsilon\}| = 0$

3. Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f(x)|\, dx < \varepsilon$$

   whenever $m(E) < \delta$.

4. Let $A, B \subseteq \mathbb{R}$ be bounded measurable sets with positive Lebesgue measure. For each $t \in \mathbb{R}$ define the function

$$g(t) = m\left(A \cap (t - B)\right)$$

   where $t - B = \{t - b : b \in B\}$.

   (a) Prove that g is a continuous function and

$$\int_{\mathbb{R}} g(t)\, dt = m(A)\, m(B).$$

   (b) Conclude that the sumset

$$A + B = \{a + b : a \in A \text{ and } b \in B\}$$

   contains a non-empty open subset of $\mathbb{R}$.

5. Let $1 \leq p < q < \infty$.

   (a) Which of the following statements (i)-(iv) are true, and which are false? Justify all the negative answers by a counterexample and all positive answers with a proof.

   (i) $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$
   (ii) $L^q(\mathbb{R}) \subseteq L^p(\mathbb{R})$
   (iii) $L^p([0,1]) \subseteq L^q([0,1])$
   (iv) $L^q([0,1]) \subseteq L^p([0,1])$

   (b) Justify your answer to the following question:
   For which values of $r \geq 1$ is it true that $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subseteq L^r(\mathbb{R})$?