Complex Analysis Qualifying Exam — Spring 2024

Committee: Peter Lambert-Cole, Lin Mu (Chair) and Jingzhi Tie

Show work and carefully justify/prove your assertions. For example, if you use a theorem that has a name, mention the name. Arrange your solutions in numerical order even if you do not solve them in that order.

1. Prove that the distinct complex numbers z_1, z_2 and z_3 form an equilateral triangle if and only if
$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

2. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r_0 < 1$, let $\overline{D}_{r_0}$ be the closed disk $|z| \le r_0$. Show that the area A of $f(\overline{D}_{r_0})$ is finite and is given by
$$A = \pi \sum_{n=1}^{\infty} n |c_n|^2 r_0^{2n}.$$
[Hint: First find a formula in terms of polar coordinates in xy-plane for the area element $dudv$ using complex analysis, where $f = u + iv$. Note that $dxdy = rdrd\theta$.]

3. Suppose f is entire and there exists $A, R > 0$ and natural number N such that $|f(z)| \le A|z|^N$ for $|z| \ge R$. Show that (i) f is a polynomial and (ii) the degree of f is at most N.

4. Computer the integral $I(b) = \int_0^{\frac{\pi}{2}} (\tan t)^{ib} dt$ for $b \in \mathbb{R}$ and $b \ne 0$. Hint: Some simple substitution will reduce the integral to a familiar form.

5. Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z \to \infty} f(z) = A$. Show that
$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

6. (a) (The maximum modulus principle) Suppose that U is a bounded domain and that $f(z)$ is a non-constant continuous function on $\bar{U}$ whose restriction to U is holomorphic. If $z_0 \in U$, show that
$$|f(z_0)| < \sup\{|f(z)| : \ z \in \partial U\}.$$
(b) Furthermore if $|f(z)|$ is constant on ∂U, then $f(z)$ has a zero in U: there exists $z_0 \in U$ for which $f(z_0) = 0$.

7. Let $G = \mathbb{D} \backslash [\frac{1}{2}, 1)$. Find a conformal map from G to the upper half plane $\mathbb{H}$. You need to write the conformal map explicitly and show that it is an one-to-one and onto map from G to the upper half plane.