M.A. Comprehensive Examination – Sp 94

Numerical Analysis: Mat603–605

Name______________________________ Student Id. No.____________

Instruction: Please work one problem from each of following four parts. Use a separate sheet of paper to do each problem and show all your work.

Part I Solution of Nonlinear Equations and Optimizations

Please do one and only one problem from problems [1],[2],and [3].

[1] Suppose that p is a root of multiplicity $m > 1$ of $f(x) = 0$. Show that the following modified Newton's method

$$p_{n+1} = p_n - \frac{mf(p_n)}{f'(p_n)}$$

gives quadratical convergence.

[2] Use Steepest Descent Method to solve

$$g(\mathbf{x}^*) = \min_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x})$$

where $g(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t A\mathbf{x} - \mathbf{x}^t\mathbf{b}$, $n = 3$, and

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Start with $\mathbf{x}^0 = (0,0,0)^t$ and do three iterations.

[3] Let F be a mapping $\mathbf{R}^n \mapsto \mathbf{R}^n$. We say F is convex over a convex domain D if

$$F(\mathbf{y}) - F(\mathbf{x}) \geq F'(\mathbf{x})(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in D$. Consider Newton's iteration

$$\mathbf{x}^{k+1} = \mathbf{x}^k - (F'(\mathbf{x}^k))^{-1}F(\mathbf{x}^k), k = 0, 1, \cdots.$$

Show that if F is convex on $\mathbf{R}^n$, then $F(\mathbf{x}^k) \geq 0$ for $k \geq 1$. Let $\mathbf{x}^*$ be the solution of $F(\mathbf{x}) = 0$. Suppose that each entry of matrix $F'(\mathbf{x})^{-1}$ is nonnegative. Further show that $\mathbf{x}^{k+1} \leq \mathbf{x}^k$ for all $k \geq 1$ and $\mathbf{x}^k \geq \mathbf{x}^*$ for all $k \geq 0$. Justify that if F is convex and each entry of $F'(\mathbf{x})^{-1}$ is nonnegative on $\mathbf{R}^n$ and $F(\mathbf{x}) = 0$ has a solution $\mathbf{x}^*$, then Newton's method converges for any initial guess $\mathbf{x}^0$.

Yim An

Part II Polynomial Interpolation and Approximation

Please do one and only one problem from problems [4],[5],and [6].

[4] Find a polynomial P_4 of degree ≤ 4 satisfying the following interpolation condition: $P_4(0) = 1, P_4(1) = -1, P_4'(0) = 2,\ P_4'(1) = 3$, and $P_4''(1) = 0$.

[5] Find the polynomial p_2^* of degree 2 solving the following continuous least square problem

$$\int_{-1}^{1} |e^x - p_2^*(x)|^2 dx = \min_{a_0,a_1,a_2} \int_{-1}^{1} |e^x - (a_0 + a_1 x + a_2 x^2)|^2 dx.$$

[6] Calculate the (2, 1) Padé approximation to e^x.

Part III Numerical Linear Algebra

Please do one and only one problem from problems [7],[8],and [9].

[7] Compute the Choleski's decomposition of Hilbert matrix

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

[8] Given an SOR iteration:

$$\mathrm{x}^{(k+1)} = (D + \omega L)^{-1}[(1 - \omega)D - \omega U]\mathrm{x}^{(k)} + \omega(D + \omega L)^{-1}\mathrm{b}$$

of solving $A\mathrm{x} = \mathrm{b}$ with $A = L + D + U$, show that ω must be in $(0, 2)$ in order to ensure the convergence of that iteration.

[9] Use Householder's transformation or Givens' rotation to reduce the following matrix to a tridiagonal matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

Part IV Numerical Solution of ODE's

Please do one and only one problem from problems [10],[11],and [12].

[10] The following is a 3-stage Runge–Kutta's method

$$\begin{cases} w_0 & = & \alpha \\ K_1 & = & hf(t_k, w_k) \\ K_2 & = & hf(t_k + h/2, w_k + K_1/2) \\ K_3 & = & hf(t_k + 3h/4, w_k + 3K_2/4) \\ w_{k+1} & = & w_k + \frac{1}{9}(2K_1 + 3K_2 + 4K_3) \end{cases}$$

Assuming that this method has local truncation error $O(h^3)$, show that this is a convergent method.

[11] Use interpolation polynomial $p_m(f,x)$ based on $t_{k+m}, t_{k+m-1}, \cdots, t_k$ to approximate the integrand $f(x, y(x))$ in the following

$$y(t_{k+m+1}) = y(t_{k+m-1}) + \int_{t_{k+m-1}}^{t_{k+m+1}} f(t, y(t))dt$$

to derive a multistep method like Adam–Bashforth's method. In particular, find the method for $m = 1$ and $m = 2$.

[12] Consider using a single step method

$$\begin{cases} \mathbf{w}_0 = \alpha, \\ \mathbf{w}_{k+1} = \mathbf{w}_k + h\phi(t, \mathbf{w}_k, h) \quad k = 0, \cdots, n-1 \end{cases}$$

to solve a system of differential equations

$$\begin{cases} Y'(t) = F(t, Y), \quad a \leq t \leq b \\ Y(a) = \alpha, \end{cases}$$

with $Y(t) = (y_1(t), \cdots, y_\ell(t))^t$, $F(t,Y) = (f_1(t; y_1, \cdots, y_\ell), \cdots, f_\ell(t; y_1, \cdots, y_\ell))^t$, and $\alpha = (\alpha_1, \cdots, \alpha_\ell)^t$. Suppose that $\phi(t, \mathbf{x}, h)$ satisfies Lipschitz's condition, i.e.,

$$\|\phi(t, \mathbf{x}, h) - \phi(t, \mathbf{y}, h)\|_2 \leq L\|\mathbf{x} - \mathbf{y}\|_2.$$

and the local truncation error $\tau_k(h) = \frac{1}{h}(Y(t_{k+1}) - Y(t_k)) - \phi(t, Y(t_k), h) = O(h^m)$ for $m > 0$, i.e., $\|\tau_k(h)\|_2 \leq Ch^m$. Show that $\max_{0 \leq k \leq n} \|Y(t_k) - \mathbf{w}_k\|_2 = O(h^m \dfrac{e^{L(b-a)} - 1}{L})$.