Algebra Preliminary Examination, April 1996

1. (a) Let V be a nonzero finite-dimensional vector space over $\mathbb{C}$, endowed with a positive definite hermitian form $<\,,\,>: V \times V \to \mathbb{C}$. Let $A : V \to V$ be a hermitian map. Show that V has an orthogonal basis consisting of eigenvectors of A.

   (b) Let $A \in M_n(\mathbb{C})$ be an hermitian matrix. Does there exist a matrix $B \in M_n(\mathbb{C})$ such that $B^n = A$? Justify your answer.

2. Let G be a group of permutations of a set S with n elements. Assume that G is transitive (i.e., for any $x, y \in S$, there exists $\sigma \in G$ such that $\sigma(x) = y$).

   (a) Show that n divides the order of G.

   (b) Suppose $n = 4$. For which integers $k \geq 1$ can such a G have a order $4k$? Justify your answer.

3. Denote by $\mathbb{F}_8$ the field with 8 elements and let $\mathbb{F}_8^+$ be its associated additive group. If R is any ring with identity 1, let R^* denote its associate multiplicative group of units. List all groups of order 8, up to isomorphism. Then identify which type occurs in each of

   (a) $(\mathbb{Z}/17\mathbb{Z})^*/(\pm 1)$,

   (b) the group of symmetries of a square,

   (c) the roots of $x^8 - 1$ in $\mathbb{C}$,

   (d) $\mathbb{F}_8^+$,

   (e) $(\mathbb{Z}/16\mathbb{Z})^*$.

4. Let k be a field. Let V be a finite-dimensional k-vector space. Let $L : V \to V$ be a linear map. If $f(x) = \sum_{i=0}^{n} a_i x^i$ is any polynomial in $k[x]$, let $f(L) = \sum_{i=0}^{n} a_i L^i$ denote the associated linear map.

   (a) Let $w \in V$. Show (directly) that there exists a nonzero polynomial $g(x) \in k[x]$ such that $g(L)(w) = 0$.

   (b) Use a) to show that there exists a nonzero polynomial $f(x) \in k[x]$ of minimal degree such that $f(L)(v) = 0$, for all $v \in V$.

   (c) Let $\lambda \in k$ be a root of $f(x)$ as in b). Show that there exists $v \in V$, $v \neq 0$, such that $L(v) = \lambda v$.

5. Let R be a commutative ring with an identity element 1 $(1 \neq 0)$.

   (a) Give the definition of a <u>maximal ideal</u> of R.

   (b) Show that R always contains a maximal ideal.

(c) Let M be an ideal of R. Show that M is a maximal ideal of R if and only if R/M is a field.

6. Let p be prime. To which finite group is the group $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ isomorphic to? Carefully justify your answer. (You may want to recall the definition of $\otimes$).

7. Let $3^{1/n}$ denote the unique real positive root of $x^n - 3$. Let $F_i := \mathbb{Q}(3^{1/2^i}), i \in \mathbb{N}$. Let $F := \mathbb{Q}(3^{1/2^i}, i \in \mathbb{N}) = \bigcup_{i\in\mathbb{N}} F_i$.

   (a) Show that F is not a finite dimensional $\mathbb{Q}$-vector space.

   (b) Fix $i \in \mathbb{N}$. Describe the group of all field automorphisms $\sigma : F_i \to F_i$. Justify your answer.

   (c) Show that the identity map $id : F \to F$ is the only field automorphism of F.

8. Consider the ideal I of $\mathbb{Z}[i]$ generated by 2 (i.e., $I = (2)$).

   (a) How many elements does the quotient ring $\mathbb{Z}[i]/I$ have? Justify your answer.

   (b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{F}_4$ are three non-isomorphic rings. Is the ring $\mathbb{Z}[i]/I$ isomorphic to any of these rings? If yes, which one? Justify your answer.