Preliminary Exam in Algebra

March 1997

Do as many problems as you can; each problem is worth 10 points. All rings have a multiplicative identity, denoted 1.

1. State and prove the Cayley-Hamilton theorem for matrices over a field.
2. How many isomorphism classes of groups of order 20 are there? Give a presentation for a group in each isomorphism class. Prove that you have a complete list.
3. In each of the following cases, determine whether the given field extension is Galois, determine the degree of the extension, and if it is Galois, describe the Galois group.
   (a) The splitting field of the polynomial $x^3 - 2$ over $\mathbb{Q}$.
   (b) The extension of $\mathbb{Q}$ formed by adjoining the real cube root of 2.
   (c) The extension $F_p(t)$ of $F_p(t^p)$, where t is an indeterminate.
4. Let P be a finite p-group. Prove that every nontrivial normal subgroup of P intersects the center nontrivially.
5. Prove that the intersection of the prime ideals of a commutative ring is equal to the largest ideal consisting of the nilpotent elements. [Hint: for one direction, use Zorn's Lemma]
6. Put the matrix
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}$$
into Jordan Canonical form.
7. Let G be the symmetric group of degree 5 (and order 120).
   (a) Describe a Sylow 2-subgroup of G
   (b) How many Sylow 5-subgroups does G have? Justify your answer.
   (c) Prove that G has a transitive permutation representation of degree 6.
8. What is meant by a left Noetherian ring? Prove that if R is a (left) Noetherian ring then every submodule of a finitely generated (left) R-module is finitely generated.