Algebra Preliminary Examination: Spring 1999

1. Let G be a finite group and let P be a Sylow p-subgroup for p a prime.
   (a) Suppose that H is a normal subgroup of G. Show that $H \cap P$ is a Sylow p-subgroup of H.
   (b) Does the same hold if H is not normal? Prove or give a counterexample.
2. Let R be a commutative ring with 1. Suppose that I is a proper ideal in R. Show there is a maximal ideal U such that $I \subseteq U \neq R$.
3. Let $\mathbb{E}$ be a subfield of the complex numbers $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Show that α is algebraic over $\mathbb{E}$ if and only if $[\mathbb{E}(\alpha) : \mathbb{E}]$ is finite.
4. Let M be an $n \times n$ matrix over the complex numbers $\mathbb{C}$ and let $V = \mathbb{C}^n$.
   (a) Show that there is a subspace W of V such that the dimension of W is 1. and $M \cdot W \subseteq W$.
   (b) Is the same true if we replace the complex numbers by the real numbers $\mathbb{R}$? Prove or give a counterexample.
5. Let $F = GF(3)$ be the field with three elements and let $R = F[x]$. Find all isomorphism classes of R-modules M such that M has exactly 81 elements and also $(x^2 - 1)^6 \cdot M = \{0\}$.
6. Let
$$M = \begin{bmatrix} 15 & -1 & -12 \\ 13 & 1 & -12 \\ 13 & -1 & -10 \end{bmatrix}.$$
   Find the minimal polynomial, characteristic polynomial and Jordan canonical form for M.
7. Let A_7 be the alternating group on 7 letters. Show that any two elements of order 5 in A_7 are conjugate.
8. Let $\mathbb{E}_n$ be the cyclotomic field $\mathbb{Q}(\zeta)$ where ζ is a primitive nth root of 1 and $\mathbb{Q}$ is the rational numbers. Find the Galois group $G(\mathbb{E}_n/\mathbb{Q})$ of the extension of $\mathbb{E}_n$ by $\mathbb{Q}$ and find all intermediate fields F with $\mathbb{Q}_n \subseteq F \subseteq \mathbb{E}_n$ for
   (a) $n = 24$ and
   (b) $n = 15$.
9. Prove that any simple group of order 1092 is isomorphic to a subgroup of A_14, the alternating group on 14 letters. (Hint: look at the Sylow 13-subgroup.)