Preliminary Exam in Algebra

January 2001

Do as many problems as you can; each problem is worth 10 points. The number of problems done **completely** will be taken into account: one correct problem is better than two half-done problems.

1. (a) State the three Sylow Theorems.
   (b) Prove that there is no simple group of order 300.

2. (a) Use the Class Equation to prove that the center of every nontrivial p-group is nontrivial.
   (b) Use part (a) to prove that every group of order p^2 (p prime) is abelian.

3. (a) Let G be a finite group acting on a set S. For $s \in S$, let G_s be the stabilizer of s, and $\mathcal{O}_s$ the orbit of s. Prove that
   $$|G| = |\mathcal{O}_s||G_s|$$
   (here $|\ |$ means cardinality).
   (b) Let *Cube* be the group of rotational symmetries of a cube. Use the formula in (a) to compute $|Cube|$.
   (c) Prove that $Cube \simeq S_4$.

4. Let $f(x) = x^5 - 1 \in \mathbb{Q}[x]$.
   (a) Find the splitting field K of $f(x)$ over $\mathbb{Q}$, and compute the degree $[K : \mathbb{Q}]$.
   (b) Compute the Galois group G of f over $\mathbb{Q}$.
   (c) Find all subgroups of G, and match them to the corresponding intermediate fields between $\mathbb{Q}$ and K.

5. (a) Prove that the Galois group of $f(x) = x^5 - 6x + 2$ over $\mathbb{Q}$ is S_5. (You may assume any structural properties of S_5 that you know.)
   (b) Explain the connection between part (a) and solving polynomial equations.

6. Let R be a ring (not assumed to have multiplicative identity) having more than one element, such that for each nonzero $a \in R$, there is a unique $b \in R$ satisfying $aba = a$.
   Prove:
   (a) R has no zero divisors;
   (b) $bab = b$ (when a and b are as above);
   (c) R has a multiplicative identity;
   (d) R is a division ring.

7. Let F be a finite field.

   (a) If F has odd order, prove that exactly half of the elements of $F^\times$ are squares, and that if α, β are non squares, then $\alpha\beta$ is a square. [Hint: you might start be doing the case $F = \mathbb{F}_p$.]

   (b) If F has even order, prove that every element of F is a square.

8. Find the Jordan canonical form over $\mathbb{C}$ of the matrix

$$\begin{bmatrix} 0 & 0 & -1 & 2 \\ 1 & 1 & 1 & -3 \\ 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Explain your reasoning.

9. (a) If A is a real normal matrix with eigenvalues, prove that A must be symmetric.

   (b) If A is any complex matrix, prove that $I + AA^*$ is nonsingular.