Real and Complex Analysis Preliminary Examination

Spring, 2002

Name____ Student Id. No._______________

Instruction: There are ten problems in total. Please work as many problems as possible. Use a separate sheet of paper to do each problem and show all your work.

[1] Suppose that f is n times continuously differentiable. Show that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{(n-1)!} f^{(n-1)}(a)(x-a)^{n-1} + \frac{1}{(n-1)!} \int_a^b f^{(n)}(t)(x-t)_+^n \, dt$$

where $(x-t)_+^n = \begin{cases} (x-t)^n, & \text{if } x \ge t \\ 0, & \text{if } x < t. \end{cases}$

[2] Suppose that a bounded function $H(x)$ defined on the real line satisfies that $H(0) = 1$ and $H(x)$ is continuously differentiable in a neighborhood of $x = 0$. Let

$$\phi(x) = \prod_{j=1}^{\infty} H(x/2^j).$$

Show that ϕ is well-defined for any real number x. Furthermore, show that ϕ is a continuous function if $H(x)$ is a continuous function.

[3] Suppose that $f, g \in L_2(-\infty, \infty)$ and f vanshes outside of a bounded closed set. Show that

$$\sum_{k=-\infty}^{\infty} 2^{-n} \left(\int_{-\infty}^{\infty} f(x) g(2^{-n}x - k) dx \right)^2 \longrightarrow 0,$$

as $n \to \infty$.

[4] Suppose that $\{f_n \in L_2(0, 2\pi), n = 1, 2, \cdots, \}$ is a sequence which is weakly convergent to $F \in L_2(0, 2\pi)$, i.e.,

$$\int_0^{2\pi} (f_n(x) - F(x)) h(x) dx \longrightarrow 0, \quad n \to +\infty,$$

for any $h \in L_2(0, 2\pi)$. Suppose also that $\|f_n\|_2 \longrightarrow \|F\|_2$ as $n \longrightarrow \infty$, where $\|f_n\|_2 = \sqrt{\int_0^{2\pi} |f_n(x)|^2 dx}$ and similar for $\|F\|_2$. Show that f_n converges to F in the L_2 norm.

[5] Suppose that ϕ is a continuous function and

$$\sum_{k=-\infty}^{\infty} |\phi(x+k)|^2 = 1, \quad \forall x \in [0,1].$$

Show that for any $\epsilon > 0$, there exists an integer K such that

$$\sum_{|k|\leq K} |\phi(x+k)|^2 \geq 1-\epsilon, \quad \forall x \in [0,1].$$

[6] Use the Rouche theorem to prove the fundamental theorem of algebra.

[7] Compute the Laurent series of function $f(z) = \frac{1}{(z^5-1)(z-3)}$ in annulus $\{z : 1 < |z| < 3\}$.

[8] Use the residue theorem to compute the following integral:

$$I = \int_0^{\infty} \frac{1}{(1+x^2)^2} dx.$$

[9] Find explicitly a conformal mapping of $\{z : 0 < \text{Im}(z) < \pi\}$ onto the unit disk.

[10] Suppose that f is a continuous function on $(-\infty, \infty)$ which vanishes outside of a bounded closed set. Define

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t) \exp(-izt) dt, \quad \forall z \in \mathbf{C}.$$

Show that $\hat{f}(z)$ is an entire function.