Preliminary Examination in Topology, January 2002

Complete answers are preferred to fragments.

1. Prove that the product of a finite number of compact spaces is compact. Is the product of two Hausdorff spaces necessarily Hausdorff?

2. Prove that every contractible space is simply connected. Give an example of a simply connected space which is not contractible.

3. Let X be the union of two copies I_1 and I_2 of the closed interval $[-1,1]$, with all points in I_1 except zero identified with the corresponding point in I_2. Is X (i) compact? (ii) Hausdorff? (iii) connected? (iv) metrizable? Justify your answers.

4. Let $a < b$ be real numbers. Show that if $f\colon [a,b] \to [a,b]$ is continuous then f has a fixed point.

5. State van Kampen's theorem. Use it to calculate the fundamental group of the one point union of two circles, $S^1 \vee S^1$.

6. Give a cell decomposition of the projective plane $\mathbf{R}P^2$, showing the attaching maps in detail, and use it to calculate the homology $H_*(\mathbf{R}P^2)$.

7. Let $(X, x_0) = S^1 \vee S^2$ be a one point union of the circle and the 2-sphere, with the common point regarded as basepoint. Find the universal cover of X. Give an example of a based map $S^2 \to X$ for which there is no homotopy to a map which avoids $S^1 \setminus \{x_0\}$. You need not prove that your map has the stated property.

8. Describe how a continuous function between topological spaces induces a homomorphism between their fundamental groups. Let T be the torus $S^1 \times S^1$. Describe the action of the group of orientation preserving homeomorphisms $T \to T$ on $\pi_1(T)$.

9. Let K be a connected finite simplicial complex such that $H_n(K)$ is a finite group for each $n > 0$, and let $f\colon K \to K$ be a self-map. Prove that f has a fixed point. State carefully any theorems you use. Give an example of a K satisfying the hypotheses, with $H_n(K) \neq 0$ for some $n > 0$.