Analysis Qualifying Exam - January 2003.

Instructions: Show all your work and prove all your assertions. To pass, you must demonstrate satisfactory knowledge of both Real Analysis and Complex Analysis.

Real Analysis Questions:

1) Show that if $E \subset \mathbb{R}$ is uncountable, then there is some $t \in \mathbb{R}$ such that both $E \cap (-\infty, t)$ and $E \cap (t, \infty)$ are uncountable.

2) Let $F = (F_1, \dots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map. Assume that $F(\vec{0}) = \vec{0}$ and that

$$\sum_{i,j=1}^{n} |\frac{\partial F_i}{\partial x_j}(\vec{0})|^2 < 1 .$$

Show there is a closed ball B centered at $\vec{0}$ such that $F(B) \subset B$.

3) Let $E \subset [0,1] \times [0,1]$ be a Lebesgue measurable set. Denote the horizontal sections of E by $E_y = \{x \in [0,1] : (x,y) \in E\}$ and the vertical sections by $E^x = \{y \in [0,1] : (x,y) \in E\}$.

Assume that $|E_y| \geq 1/2$ for every $0 \leq y \leq 1$, where $|A|$ denotes the Lebesgue measure of a measurable set A.

Show that the set $H = \{x \in [0,1] : |E^x| \geq 1/4\}$ is Lebesgue measurable, and that $|H| \geq 1/3$.

4) Let $E \subset \mathbb{R}$ be an open set of finite measure. Define $f(x) = |E \cap (E + x)|$ for $x \in \mathbb{R}$. Here $E + x = \{z + x;\ z \in E\}$ and $|A|$ denotes the Lebesgue measure of a set A.

Show that $\lim_{x \to \infty} f(x) = 0$ and that f is uniformly continuous on $\mathbb{R}$.

5) Let $(H, \langle , \rangle)$ be a real Hilbert space; write $\|x\|^2 = \langle x, x \rangle$. Suppose $K \subset H$ is a closed convex subset, and that $x \in H \backslash K$.

a) Show there is a vector $z \in K$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in K$.

b) Show there are a unit vector $u \in H$ and a constant C such that $\langle u, x \rangle < C \leq \langle u, y \rangle$ for all $y \in K$.

Complex Analysis Questions:

6) Exhibit a conformal map from the strip $\{z \in \mathbb{C} : 0 < \text{Im}(z) < 1\}$ onto the unit disc $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

7) Compute $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$, where $a > 0$.

8) Show there is a $T > 0$ such that $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(nz)$ converges uniformly to a function analytic on $\{z \in \mathbb{C} : -T < \operatorname{Im}(z) < T\}$, and find the largest such T.

9) For each integer $n \geq 1$, let $P_n(z) = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots + \frac{1}{n!}z^n$. Show that for all sufficiently large n, $P_n(z)$ has no zeros in $D(0, 10) = \{z \in \mathbb{C} : |z| < 10\}$, and $P_n(z) - 1$ has exactly 3 zeros there.

10) Suppose $f(z)$ is an entire function, and that for each $a \in \mathbb{C}$ at least one coefficient $c_n = c_n(a)$ in the Taylor expansion $f(z) = \sum_{n=0}^{\infty} c_n(a)(z - a)^n$ is 0. Show that $f(z)$ must be a polynomial.