# Numerical Analysis Preliminary Examination

**Spring 2003**

Name________________________________________ Student Id. No.________________

**Instruction:** *The following are ten problems in total. Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours.*

[1] Suppose that $f \in C^2(\mathbf{R})$ is increasing and $f'' > 0$. Suppose that $f$ has a zero. Show that the Newton iteration for computing the zero of $f$ will converge for any starting point.

[2] Let $A$ be a strictly diagonally dominant. Show that the Gauss-Jacobi iterative method for solving $A\mathbf{x} = \mathbf{b}$ converges for any initial guess.

[3] Let $\mathbf{x}$ and $\tilde{\mathbf{x}}$ be the solution of two linear systems: $A\mathbf{x} = \mathbf{b}$ and $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. Show that

$$\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq \text{cond}(A)\frac{\|\mathbf{b} - \tilde{\mathbf{b}}\|}{\|\mathbf{b}\|}$$

where $\text{cond}(A) = \|A\|\|A^{-1}\|$ stands for the condition number of $A$. If the cond $(A)$ is $10^{10}$ and a hypothetical computer does arithmetic operation to 15 significant digits, how many significant digits can you expect from the solution using this computer? Explain your reasoning.

[4] Recall that for any nonzero vector $v$ of size $n \times 1$, $I - 2\frac{vv^T}{\|v\|^2}$ is called a Householder matrix, where $I$ is the $n \times n$ identity matrix. Show that there exists a sequence of Householder matrices $H_1, \ldots, H_n$ which converts any matrix $A$ into a lower triangular matrix $L$, that is, $H_n \cdots H_1 A = L$.

[5] Recall the $2 \times 2$ Givens rotation matrix is

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Use Givens rotation matrices to reduce the following matrix to a tridiagonal matrix:

$$\begin{bmatrix} 7 & 3 & -2 \\ 3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}.$$

[6] Let $p_n(x) = \sum_{i=0}^{n} c_i B_i^n(x)$ be a polynomial in B-form with respect to $[a, b]$. Here, $B_i^n(x) = \binom{n}{i}\left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{n-i}$ is defined on the interval $[a, b]$. Similarly, let

$q_n(x) = \sum_{i=0}^{n} d_i \tilde{B}_i^n(x)$ with $\tilde{B}_i^n(x) = \binom{n}{i} \left(\frac{x-b}{c-b}\right)^i \left(\frac{c-x}{c-b}\right)^{n-i}$ defined on $[b, c]$. Derive the conditions that ensure

$$\frac{d^r}{dx^r} p_n(b) = \frac{d^r}{dx^r} q_n(b), \quad \forall r = 0, 1, 2.$$

[7] Suppose that $\phi_n, n = 0, 1, 2, \cdots$, are orthonormal polynomials over interval $[a, b]$. Let $\{x_i^{(n)}, i = 1, \ldots, n\}$ be the roots of polynomial $\phi_n$ of degree $n$ for $n \geq 1$. Define the well-known Gaussian quadrature by

$$G_n(f) := \sum_{i=1}^{n} f(x_i^{(n)}) a_i$$

with $a_i = \int_a^b \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x - x_j^{(n)})}{(x_i^{(n)} - x_j^{(n)})} dx, i = 1, \cdots, n$. Show that $G_n(p) = \int_a^b p(x) dx$ for all polynomial $p$ of degree $\leq 2n - 1$.

[8] Show that the following Runge–Kutta's method has local truncation error $O(h^3)$:

$$\begin{aligned}
K_1 &= hf(x_k, y_k) \\
K_2 &= hf(x_k + h/3, y_k + K_1/3) \\
K_3 &= hf(x_k + 2h/3, y_k + 2K_2/3) \\
y_{k+1} &= y_k + \frac{1}{4}(K_1 + 3K_3)
\end{aligned}$$